Solution Sheet 13

Due date: **July 24, 2017, 15:30**. Discussion of solutions: July 24, 2017.

Problem 37. **no points**

Let $n > 2k$. Prove that for each maximum, intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ there is an $x \in [n]$ contained in all members of the family $\mathcal{F}$, i.e., x is fixed by the family.

Hint: Consider the proof of the Erdős-Ko-Rado Theorem.

Solution.

Let $\mathcal{F}$ denote a maximum, intersecting in $\binom{[n]}{k}$. Then $|\mathcal{F}| = \binom{n-1}{k-1}$ by the Erdős-Ko-Rado Theorem. Following the proof of the Erdős-Ko-Rado Theorem, we see that each cyclic permutation of $[n]$ meets (exactly) k members of $\mathcal{F}$.

Consider a cyclic permutation σ and the sets $A_1, \ldots, A_k$ met by σ. We claim that there is an $x \in [n]$ contained in all A_i (see Figure 1 left). Indeed let $a_1, \ldots, a_k$ denote the elements of A_1 labeled according to the ordering along σ. Then for each $i \in \{2, \ldots, k\}$ there is some j such that A_i contains exactly one of a_j and a_{j+1}. We say that A_i *ends in the gap* $a_j a_{j+1}$. Observe that for each "gap" in A_1 there is exactly one of the sets $A_2, \ldots A_k$ which "ends" there, since $n > 2k$ and the sets are intersecting. If for each gap $a_j a_{j+1}$ the set which ends in this gap contains a_{j+1} and not a_j ("ends from the right"), then $x = a_k$ is contained in all the sets. Otherwise let j_0 denote the smallest $j \in [k]$ such that there is some set from $A_2, \ldots, A_k$ which contains a_j and not a_{j+1} ("ends from the left"). Observe that if some A_i ends in the gap $a_j a_{j+1}$ (for some $j \leq k-2$) and contains a_j but not a_{j+1} ("ends from the left"), then for the next "gap" $a_{j+1}a_{j+2}$ in A_1 there is no set from $A_2, \ldots, A_k$ which does not contain a_{j+1} but contains a_{j+2} ("ends from the right"), again since $n > 2k$ and the sets are intersecting. Hence for each $j \geq j_0$ we have that the set which ends in gap $a_j a_{j+1}$ contains a_j and not a_{j+1} ("ends from the left") and for each $j < j_0$ we have that the set which ends in gap $a_j a_{j+1}$ contains a_{j+1} and not a_j ("ends from the right"). This shows that $x = a_{j_0}$ is contained in all sets $A_1, \ldots, A_k$.

Clearly we can transform σ into any other cyclic permutation π using only transpositions of consecutive elements not equal to x. Let π be a permutation obtained by applying one such transposition to σ. We claim that all the members of $\mathcal{F}$ met by π still contain x. This shows that all members of F contain x. To prove the claim consider the transposition (ab) applied to σ ($a, b \neq x$). Then all but at most one of the sets $A_1, \ldots, A_k$ contain either both a and b or neither. Suppose, without loss of generality, that A_j contains a but not b. Then π meets all the sets A_i for $i \neq j$. If $n > 2k$, then the only set that is met by π and which intersects all A_i with $i \neq j$ is $A' = (A_j \setminus \{a\}) \cup \{b\}$, see Figure 1 (middle and right). Thus $A' \in \mathcal{F}$. But $x \in A'$ since $x \in A$ and $x \neq a$. Hence x is contained in all members of $\mathcal{F}$ met by π. This proves the claim.

Note that $n > 2k$ is necessary, since if $n = 2k$ and a is the last element in A_k in σ, then the k-set starting with a in π and going clockwise along π ends right before x, but does not contain x, and intersects all the sets $A_1, \ldots, A_{k-1}$. □

Problem 38. **no points**

Let n be odd. Prove that there are exactly two maximum antichains in B_n.

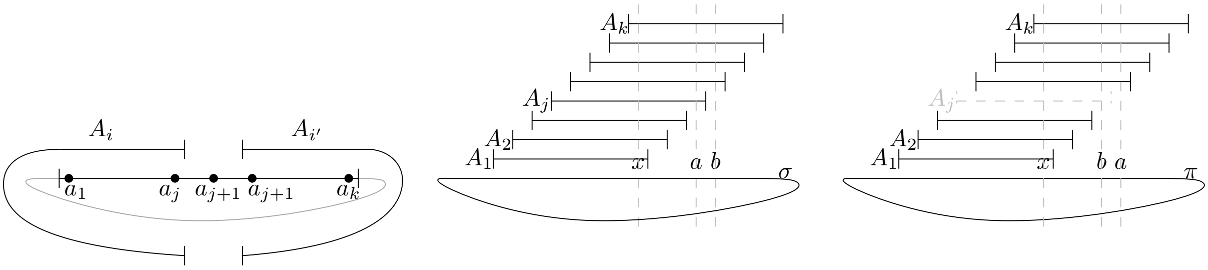
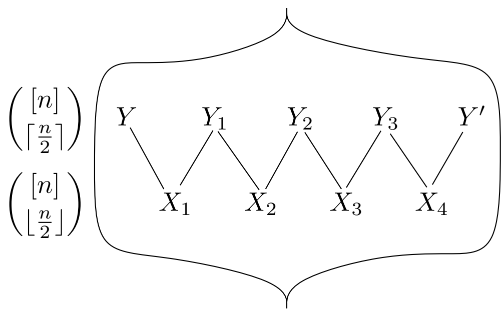


Figure 1: There is only one set met by the permutation π which intersects all the sets A_i, $i \neq j$, namely $(A_j \setminus \{a\}) \cup \{b\}$, since $n > 2k$.

Hint: If an antichain contains sets from $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and from $\binom{[n]}{\lceil \frac{n}{2} \rceil}$, find $\pi \in S_n$ that meets no set of this antichain.

Solution.
Clearly $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ as well as $\binom{[n]}{\lceil \frac{n}{2} \rceil}$ are maximum antichains by Sperner's theorem.
We show that there is no other maximum antichain. We know from the lecture that each maximum antichain is contained in $\binom{[n]}{\lfloor \frac{n}{2} \rfloor} \cup \binom{[n]}{\lceil \frac{n}{2} \rceil}$. Consider an antichain $\mathcal{F}$ which contains a set X from $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and a set Y in $\binom{[n]}{\lceil \frac{n}{2} \rceil}$. We will find a permutation which does not meet π and argue why $\mathcal{F}$ is not maximum. To this end we will find two related sets which are both not contained in $\mathcal{F}$, one from each middle layer. There is a set Y' in $\binom{[n]}{\lceil \frac{n}{2} \rceil}$ not in the antichain $\mathcal{F}$ (for instance one that is related to X). Let $X_1, \dots, X_k \in \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and $Y_1, \dots, Y_{k-1} \in \binom{[n]}{\lceil \frac{n}{2} \rceil}$ be distinct sets such that $X_1 \subset Y$, $X_k \subset Y'$ and $X_i \subset Y_i$ and $X_{i+1} \subset Y_i$ for $i \in [k-1]$ (i.e. the sets X_i, Y_i form a path from Y to Y').



Then $X_1 \notin \mathcal{F}$, since $Y \in \mathcal{F}$ (and $\mathcal{F}$ is an antichain). If $X_k \notin \mathcal{F}$ then we are done since $Y' \notin \mathcal{F}$. Otherwise there is an $i \in [k-1]$ with $X_i \notin \mathcal{F}$ and $X_{i+1} \in \mathcal{F}$. Hence $Y_i \notin \mathcal{F}$.
Let $y \in Y_i \setminus X_i$ and consider a permutation π with the prefix X_i and y on position $|X_i|+1$. This permutation does not meet any of the sets in $\mathcal{F}$.

Following the proof of Sperner's Theorem we calculate

$$\sum_{A\in\mathcal{F}} |A|!(n-|A|)! = \sum_{A\in\mathcal{F}} \{\pi \mid \pi \text{ meets } A\} < n!$$

$$\Rightarrow \qquad \sum_{A\in\mathcal{F}} \frac{1}{\binom{[n]}{|A|}} < 1$$

$$\Rightarrow \qquad \sum_{A\in\mathcal{F}} \frac{1}{\binom{[n]}{\lceil \frac{n}{2} \rceil}} < 1$$

$$\Rightarrow \qquad |\mathcal{F}| < \binom{[n]}{\lceil \frac{n}{2} \rceil}$$

Hence $\mathcal{F}$ is not a maximum antichain. □

Problem 39. **no points**

Let $m_1, \ldots, m_k \in \mathbb{N}$.

(a) Prove that the width of $B = B(m_1, m_2, \ldots, m_k)$ equals the size of the antichain $\{x \in B \mid \text{rank}_P(x) = \left\lceil \frac{\text{rank}(B)}{2} \right\rceil\}$.

Hint: Consider certain chains.

(b) Calculate the width of $B(m_1, m_2, \ldots, m_k)$.

Hint: Do not expect a nice formula, you may need the PIE.

Solution.

(a) Let $A = \{x \in B \mid \text{rank}_B(x) = \lceil \frac{\text{rank}(B)}{2} \rceil\}$. There is a symmetric chain decomposition $\mathcal{C}$ of $B = B(m_1, m_2, \ldots, m_k)$ due to the lecture. By definition of a decomposition each element from A is contained in a chain from $\mathcal{C}$, but each chain contains at most one element from A (since A is an antichain). Furthermore each symmetric chain contains one element from A, otherwise the sum of ranks of the minimum and the maximum of the chain is either smaller or larger than rank(B).

Thus $|A| = |\mathcal{C}|$ and hence any antichain larger than A would contain two elements from at least one of the chains in $\mathcal{C}$ by pigeonhole principle. But this is not possible.

Hence A is a largest antichain and thus the width of B equals $|A|$.

(b) Let $M = \sum_{i=1}^{k} m_i$. Then $\text{rank}(\{x_1, \ldots, x_k\}) = x_1 + \ldots + x_k + 1$ and $\text{rank}(B) = M + 1$, since $B = C_{m_1} \times \cdots \times C_{m_k}$ due to the lecture and $\text{rank}(C_{m_i}) = m_i + 1$.

A middle rank of B is $A = \{x \in B \mid \text{rank}_B(x) = \lceil \frac{\text{rank}(B)}{2} \rceil = \lfloor \frac{M}{2} \rfloor + 1\}$ which is given by the multisets with multiplicities $x_1, \ldots, x_k$ where $x_1 + \cdots + x_k + 1 = \lfloor \frac{M}{2} \rfloor + 1$ and $x_i \le m_i$ for all $i \in [k]$. This is exactly the number of $\lfloor \frac{M}{2} \rfloor$-combinations of a multiset with repetitions numbers m_i.

By Theorem 2.6 (multiset PIE) the number of such combinations equals

$$\sum_{S\subseteq[k]}(-1)^{|S|}\binom{k-1+\lfloor\frac{M}{2}\rfloor-\sum_{i\in S}(m_i+1)}{k-1}.$$

□