Solution Sheet 6

Due date: **June 6, 2017, 15:30**. Discussion of solutions: June 12, 2017.

Problem 16. **6 points**

For $n \geq 0$ let a_n denote the number of well-formed parenthesis expressions with n pairs of parenthesis. Here $a_0 = 1$ counts the (unique) empty expression.

(a) Prove that for each $n \geq 1$ we have $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$.

(b) Prove that there are exactly a_n monotone lattice paths from $(0,0)$ to (n,n) which do not pass through points (x,y) with $y > x$ (i.e., do not walk above the diagonal).

Solution.

(a) Let P denote the set of well-formed parenthesis expressions with n pairs of parenthesis. For $k \in [n]$ let $P_k \subseteq P$ denote the subset of those expressions π where $2k$ is the first position in π where the number of opening brackets equals the number of closing brackets. Then $P = P_2 \cup \cdots \cup P_{2n}$ and the P_k's are pairwise disjoint. Now for some $\pi \in P_k$ let π' denote the expression formed by the parenthesis at position 2 up to $2k-1$ and let let π'' denote the expression formed by the parenthesis at positions greater than k. Then π' is a well formed parenthesis expression with $k-1$ pairs of parenthesis while π'' is such an expression with $n-k$ pairs of parenthesis. Moreover for different $\pi \in P_i$ at least one of π' or π'' is different. This shows that $|P_i| \leq a_{k-1}a_{n-k}$ and hence $|P| = \sum_{k=1}^{n} |P_k \leq \sum_{k=1}^{n} a_{k-1}a_{n-k}$.

The other way round consider any well-formed parenthesis expression π' with $k-1$ pairs of parenthesis and another well-formed parenthesis expression π'' with $n-k$ pairs of parenthesis. Enclosing π' in another pair of parenthesis gives a well-formed parenthesis expression in P_k. Moreover this expression differs for different choice of π' or π''. This shows that $|P_k| \geq a_{k-1}a_{n-k}$ and hence $|P| \geq \sum_{k=1}^{n} a_{k-1}a_{n-k}$. Altogether $|P| = \sum_{k=1}^{n} a_{k-1}a_{n-k}$.

(b) Let P^n denote the set of well-formed parenthesis expressions with n pairs of parenthesis and let L^n denote the set of lattice paths from $(0,0)$ to (n,n) that do not walk above the diagonal. We claim that mapping opening brackets to right steps and closing brackets to up steps establishes a bijection between P^n and L^n. Indeed, if $\pi \in P^n$, then there are n right steps and n up steps, so the lattice path $f(\pi)$ ends in (n,n). Moreover, at each position of $f(\pi)$ the number of right steps is at least as large as the number of up steps. Therefore $f(\pi) \in L^n$. The mapping f is clearly injective and an immediate inverse. This shows that f is a bijection and thus $|L^n| = |P^n| = a_n$. □

Problem 17. **6 points**

(a) Prove that $\binom{-\frac{1}{2}}{k} = (-1)^k 2^{-2k} \binom{2k}{k}$ for each $k \in \mathbb{N}_0$.

(b) Prove that $(1-4x)^{-\frac{1}{2}}$ is the generating function of the sequence $a_n = \binom{2n}{n}$.

(c) Prove the identity $2^{2n} = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$ for all $n \in \mathbb{N}_0$.

Hint: parts a), b) and c) are related.

Solution.

(a) We will prove this by induction on k. If $k = 0$, then $\binom{-\frac{1}{2}}{0} = 1 = (-1)^0 2^0 \binom{0}{0}$. If $k > 0$, then

$$\binom{-\frac{1}{2}}{k} = \binom{-\frac{1}{2}}{k-1}\frac{\frac{1}{2}-k}{k} \overset{IH}{=} (-1)^{k-1}2^{-2(k-1)}\binom{2k-2}{k-1}\frac{\frac{1}{2}-k}{k}$$
$$= (-1)^{k-1}2^{-2(k-1)}\binom{2k-2}{k-1}\frac{2k\,(2k-1)}{k^2}\frac{-1}{2^2} = (-1)^k 2^{-2k}\binom{2k}{k}.$$

(b) The following calculation is sufficient

$$(1-4x)^{-\frac{1}{2}} \overset{(*)}{=} \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-4x)^n \overset{(a)}{=} \sum_{n=0}^{\infty}(-1)^n 2^{-2n}\binom{2n}{n}(-1)^n 2^{2n}x^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

Here $(\star)$ is an application of Newton's Binomial Theorem.

(c) We consider $(1-4x)^{-1}$. On the one hand Newton's Binomial Theorem yields

$$(1-4x)^{-1} = \sum_{n=0}^{\infty}\underbrace{\binom{-1}{n}}_{=(-1)^n}(-4)^n x^n = \sum_{n=0}^{\infty}2^{2n}x^n.$$

On the other hand we can square the power series from part (b) and obtain

$$(1-4x)^{-1} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n \cdot \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}x^n.$$

Comparing the coefficients of both power series representations yields the identity. □

Problem 18. **6 points**

Find a closed form of the generating function for the sequences a_n and b_n below.

(a) Let a_n be the number of ways you can pay n cents with dimes (10 cents), nickels (5 cents), pennies (1 cent), and quarters (25 cents). The order of coins doesn't matter.

(b) Let b_n be the number of n-combinations of $\{\infty \cdot A, \infty \cdot B, \infty \cdot C, 4 \cdot D\}$ where A occurs an odd number of times, B appears at least 10 times, and the number of C's is a multiple of 3 or 5.

Find a closed form of the exponential generating function of the sequence c_n below.

(c) Let c_n be the number of strings of length n with letters from $\{W, X, Y, Z\}$ if there is at least one X, the number of Y's is odd, and the number of Z's is either 1 or 2.

Solution.

(a) Let D_n, N_n, P_n, and Q_n denote the number of ways to pay n cents with only dimes, only nickels, only pennies, or only dimes, respectively. Then we have

$$D_n = \begin{cases} 1, & n \text{ divisible by } 10, \\ 0, & \text{otherwise}, \end{cases} \qquad N_n = \begin{cases} 1, & n \text{ divisible by } 5, \\ 0, & \text{otherwise}, \end{cases}$$

$$P_n = \begin{cases} 1, & n \text{ divisible by } 1, \\ 0, & \text{otherwise}, \end{cases} \qquad Q_n = \begin{cases} 1, & n \text{ divisible by } 25, \\ 0, & \text{otherwise}. \end{cases}$$

Let F_D, F_N, F_P, and F_Q denote the generating functions of respective sequences. Then

$$F_D(x) = \sum_{n=0}^{\infty} x^{10n} = \frac{1}{1-x^{10}}, \qquad F_N(x) = \sum_{n=0}^{\infty} x^{5n} = \frac{1}{1-x^{5}},$$

$$F_P(x) = \sum_{n=0}^{\infty} x^{n} = \frac{1}{1-x}, \qquad F_Q(x) = \sum_{n=0}^{\infty} x^{25n} = \frac{1}{1-x^{25}}.$$

Let F denote the generating function for a_n. Since each way to pay n cents can be written as a union of ways to pay with only dimes, only nickels, only pennies, or only quarters we have

$$\begin{aligned} F(x) &= F_D(x) \cdot F_N(x) \cdot F_P(x) \cdot F_Q(x) \\ &= \frac{1}{(1-x^{10})(1-x^5)(1-x)(1-x^{25})}. \end{aligned}$$

(b) Let A_n, B_n, C_n, and D_n denote the number of n combinations satisfying the conditions in the exercise that consist of only A's, only B's, only C's, or only D's, respectively. Then we have

$$A_n = \begin{cases} 1, & n \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \qquad B_n = \begin{cases} 1, & n \geq 10, \\ 0, & \text{otherwise}, \end{cases}$$

$$C_n = \begin{cases} 1, & n \text{ divisible by 3 or 5}, \\ 0, & \text{otherwise}, \end{cases} \qquad D_n = \begin{cases} 1, & n \leq 3, \\ 0, & \text{otherwise}. \end{cases}$$

Let F_A, F_B, F_C, and F_D denote the generating functions of respective sequences. Then

$$\begin{aligned} F_A(x) &= \sum_{n=0}^{\infty} x^{2n+1} = x \cdot \sum_{n=0}^{\infty} x^{2n} = \frac{x}{1-x^2}, \\ F_B(x) &= \sum_{n=10}^{\infty} x^{n} = \sum_{n=0}^{\infty} x^{n+10} = x^{10} \cdot \sum_{n=0}^{\infty} x^{n} = \frac{x^{10}}{1-x}, \\ F_C(x) &= \sum_{n=0}^{\infty} x^{3n} + \sum_{n=0}^{\infty} x^{5n} - \sum_{n=0}^{\infty} x^{15n} = \frac{1}{1-x^3} + \frac{1}{1-x^5} - \frac{1}{1-x^{15}}, \\ F_D(x) &= 1 + x + x^2 + x^3 + x^4. \end{aligned}$$

Let F denote the generating function for b_n. Since each n-combination of $\{\infty \cdot A, \infty \cdot B, \infty \cdot C, 4 \cdot D\}$ can be written as a union of combinations with only A's, only B's, only C's, or only D's we have

$$F(x) = F_A(x) \cdot F_B(x) \cdot F_C(x) \cdot F_D(x)$$

(c) Let W_n, X_n, Y_n, and Z_n denote the number of strings of length n satisfying the conditions in the exercise that consist of only W's, only X's, only Y's, or only Z's, respectively. Then we have

$$W_n = 1,\ n \geq 0 \qquad X_n = \begin{cases} 1, & n \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$Y_n = \begin{cases} 1, & n \text{ odd}, \\ 0, & \text{otherwise,} \end{cases} \qquad Z_n = \begin{cases} 1, & n = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let F_W, F_X, F_Y, and F_Z denote the generating functions of respective sequences. Then

$$\begin{aligned} F_W(x) =& \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \\ F_X(x) =& \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1, \\ F_Y(x) =& \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh(x), \\ F_Z(x) =& x + \frac{x^2}{2}. \end{aligned}$$

Let F denote the generating function for c_n. Since each string can be written as a combination of strings with only W's, only X's, only Y's, or only Z's we have

$$F(x) = F_W(x) \cdot F_X(x) \cdot F_Y(x) \cdot F_Z(x) = e^x(e^x - 1)\sinh(x)\left(x + \tfrac{x^2}{2}\right).$$

□