Solution Sheet 9

Due date: **June 26, 2017, 15:30**. Discussion of solutions: June 26, 2017.

Problem 25. **6 points**

Torsten proved in the lecture that the number of partitions of n into odd numbers is equal to the number of partitions of n into distinct numbers using a bijection.

(a) Give a proof of this fact using generating functions.

(b) Generalise this: For $l \in \mathbb{N}$, show that the number of partitions of n where no part is a multiple of l equals the number of partitions of n where no part is repeated l or more times.

Hint: Use generating functions as in Euler's Theorem (Theorem 4.5).

Solution.

(b) Consider some fixed $l \in \mathbb{N}$. For fixed k the generating function for the number of partitions where each part is repeated less than l times and has size exactly k is given by

$$\sum_{j=0}^{l-1} x^{kj} = \frac{1 - x^{kl}}{1 - x^k}.$$

Since any partition where each part is repeated less than l times is a union of smaller partitions (with fixed part size) the generating function of these partitions is given by

$$F(x) = \prod_{k \geq 0} \frac{1 - x^{kl}}{1 - x^k}.$$

Next, for fixed k the generating function for the number of partitions where no part is a multiple of l and each part has size exactly k is given by

$$\begin{cases} 1, & \text{if } l \text{ divides } k, \\ \sum_{j \geq 0} x^{kj} = \frac{1}{1 - x^k}, & \text{otherwise.} \end{cases}$$

Since any partition where where no part is a multiple of l is a union of smaller partitions (with fixed part size) the generating function of these partitions is given by

$$G(x) = \prod_{k \text{ not mult. of } l} \frac{1}{1 - x^k} = \prod_{k \geq 0} \prod_{j=1}^{l-1} \frac{1}{1 - x^{kl+j}}.$$

Now observe

$$F(x) = \prod_{k \geq 0} \frac{1 - x^{kl}}{1 - x^k} = \frac{\prod_{k \geq 0} 1 - x^{kl}}{\prod_{k \geq 0} 1 - x^k} = \prod_{k \geq 0} \prod_{j=1}^{l-1} \frac{1}{1 - x^{kl+j}} = G(x).$$

□

Problem 26. **6 points**

For $n \geq 0$ let $p_{\text{nc}}(n)$ be the number of non-crossing partitions of $[n]$ with parts of size at most 2 each.

(a) Find $p_{\text{nc}}(0)$, $p_{\text{nc}}(1)$, $p_{\text{nc}}(2)$ and a recurrence relation for $\{p_{\text{nc}}(n)\}_{n\geq 0}$.

(b) Find a closed form expression for the ordinary generating function of $\{p_{\text{nc}}(n)\}_{n\geq 0}$.

Hint: Find a defining equation for the ordinary generating function of $\{p_{\text{nc}}(n)\}_{n\geq 0}$.

Solution.

Let us call a non-crossing partition of $[n]$ with parts of size at most 2 simply a *good partition*.

(a) We clearly have that $p_{\text{nc}}(0) = p_{\text{nc}}(1) = 1$ since, for $n = 0$ there is only the empty good partition of $[n]$ and for $n = 1$ the only good partition of $[n]$ is $X = \{\{n\}\}$. We claim that for $n \geq 2$ we have

$$p_{\text{nc}}(n) = p_{\text{nc}}(n-1) + \sum_{k=0}^{n-2} p_{\text{nc}}(k)\, p_{\text{nc}}(n-2-k).$$

For $n \geq 2$ consider any good partition $X = \{X_1, \ldots, X_\ell\}$ of $[n]$. Without loss of generality we have $n \in X_\ell$.

Case 1: If $|X_\ell| = 1$, then $X_\ell = \{n\}$. Hence we can map X onto $X - X_\ell$, which is a good partition of $[n-1]$. Conversely, every good partition X' of $[n-1]$ is the image of only one good partition of $[n]$, obtained by adding $\{n\}$ to X', and this good partition has n in a singleton set.

Case 2: If $|X_\ell| = 2$, then $X_\ell = \{n, k\}$ for some element $k \in [n-1]$. As X is non-crossing, for $i = 1, \ldots, \ell-1$ we have $X_i \subseteq [k-1]$ or $X_i \subseteq \{k+1, \ldots, n-1\}$. We map X onto two good partitions X', X'', where X' consists of all sets X_i completely contained in $[k-1]$ and X'' consists of all sets X_i completely contained in $\{k+1, \ldots, n-1\}$. Clearly X' is a good partition of $[k-1]$. Relabeling each element $j \in \{k+1, \ldots, n-1\}$ into $j-k$ shows that X'' is a good partition of $[n-1-k]$. Moreover, for every $k \in [n-1]$ every ordered pair (X', X'') of a good partition of $[k-1]$ and a good partition of $[n-1-k]$ is the image of exactly one good partition X of $[n]$, where $X = X' \cup X'' \cup \{n, k\}$ after all elements j in X'' are relabeled into $j+k$.

The two bijections established above prove that for $n \geq 2$ we have

$$p_{\text{nc}}(n) = p_{\text{nc}}(n-1) + \sum_{k=1}^{n-1} p_{\text{nc}}(k-1)p_{\text{nc}}(n-1-k) = p_{\text{nc}}(n-1) + \sum_{k=0}^{n-2} p_{\text{nc}}(k)p_{\text{nc}}(n-2-k).$$

(b) Let $F(x) = \sum_{n\geq 0} p_{\text{nc}}(n)x^n$ be the ordinary generating function for $\{p_{\text{nc}}(n)\}_{n\geq 0}$. Using the recursion in the previous item we get

$$
\begin{aligned}
& p_{\mathrm{nc}}(n) = p_{\mathrm{nc}}(n-1) + \sum_{k=0}^{n-2} p_{\mathrm{nc}}(k) p_{\mathrm{nc}}(n-2-k) \qquad \forall n \geq 2 \\
\Rightarrow \qquad & \sum_{n\geq 2} p_{\mathrm{nc}}(n) x^n = \sum_{n\geq 2} p_{\mathrm{nc}}(n-1) x^n + \sum_{n\geq 2} \left(\sum_{k=0}^{n-2} p_{\mathrm{nc}}(k) p_{\mathrm{nc}}(n-2-k) \right) x^n \\
\Rightarrow \qquad & 1 + x + F(x) = 1 + x + x \sum_{n\geq 1} p_{\mathrm{nc}}(n) x^n + \sum_{n\geq 0} \left(\sum_{k=0}^{n} p_{\mathrm{nc}}(k) p_{\mathrm{nc}}(n-k) \right) x^{n+2} \\
\overset{p_{\mathrm{nc}}(0)=p_{\mathrm{nc}}(1)=1}{\Rightarrow} \qquad & F(x) = 1 + xF(x) + x^2 F^2(x),
\end{aligned}
$$

where in the last equality we used the way how ordinary generating functions multiply.

Now the equation $F(x) = 1 + xF(x) + x^2F^2(x)$ can be solved like a quadratic equation and we get two solutions

$$
F_1(x) = \frac{(1-x) + \sqrt{(1-x)^2 - 4x^2}}{2x^2}, \quad F_2(x) = \frac{(1-x) - \sqrt{(1-x)^2 - 4x^2}}{2x^2}.
$$

Due to the calculations above, $F(x)$ is necessarily equal to one of these two functions, at least in its radius of convergence. Recall from the lecture that the Bell number B_k denotes the total number of partitions of $[k]$ and the generating function for these numbers is e^{e^x-1}. In particular the generating function for B_k has positive radius of convergence. Clearly $0 \leq p_{\mathrm{nc}}(k) \leq B_k$ and hence the generating function for the numbers $p_{\mathrm{nc}}(k)$ has positive radius of convergence by the comparison test for infinite series.

To decide which of the functions F_1 and F_2 is correct we consider $x = 0$. The correct answer needs to have a power series expansion about $x = 0$ which takes the value $p_{\mathrm{nc}}(0) = 1$ there. We have

$$
F_1(x) = \frac{(1-x) + \sqrt{(1-x)^2 - 4x^2}}{2x^2} \longrightarrow \infty \qquad (x \to 0).
$$

This shows that F_1 can't be the solution and thus

$$
F(x) = F_2(x) = \frac{(1-x) - \sqrt{(1-x)^2 - 4x^2}}{2x^2}.
$$

□

Problem 27. **6 points**

Let Q be the standard Young tableaux created from a permutation π by the Robinson-Schensted-Correspondence. Show that the row of Q containing $i+1$ is strictly below the row of Q containing i if and only if $\pi^{-1}(i+1) < \pi^{-1}(i)$.

Solution.

Suppose first that $\pi^{-1}(i+1) < \pi^{-1}(i)$. Consider the initial point set X and a point set X' at some time during the algorithm. We shall prove (by induction on $|X|$) the following claim.

Claim. *If there is a point $(j,i) \in X'$, then there is also a point $(j', i+1) \in X'$ and $j' < j$.*

If $|X| = 2$, then $X = \{(1,2),(2,1)\}$ and $i = 1$, $j = 2$, $j' = 1$. There is one shadow line in the first phase. This line contains both points and has a concave bend at $(2,2)$. In particular in the next phase there is no point of the form $(\cdot, i) = (\cdot, 1)$. This gives an induction basis.
Suppose that $|X| > 2$ and consider the first phase. Let L_i and L_{i+1} denote the shadow line through $(\pi^{-1}(i), i)$ and $(\pi^{-1}(i+1), i+1)$ respectively. Observe that if $(\pi^{-1}(i), i)$ is a minimum, then $(\pi^{-1}(i+1), i+1)$ is either already taken by a shadow line or a minimum as well. Hence either $L_i = L_{i+1}$ or L_i is strictly to the right of L_{i+1}. This shows that the line L_{i+1} has a concave bend $(j', i+1)$ with $\pi^{-1}(i+1) < j' \leq \pi^{-1}(i)$. Moreover the line L_i either leaves at height i (so i is added to the tableaux in this phase) and there is no point of the form (j,i) in any set X' of the following phases, or it has a concave bend at some point (j,i) with $j > \pi^{-1}(i)$. The claim clearly holds in the first case. So consider the second case. Here the point set $\bar{X}$ in the second phase has points (j,i) and $(j', i+1)$ with $j' < j$. By considering this point set as new initial point set we can apply induction. Therefore any point set X' in some later phase satisfies the claim. This proves the claim.

Now, given the claim, consider the phase when entry i is added to the tableaux. By the claim there exists a point $(j', i+1)$ with $j' < j$ in this phase. Now the line through $(j', i+1)$ can't leave in this phase, since it has a concave bend somewhere between j' and j. Therefore the entry $i+1$ is added to the tableaux in some later phase and thus strictly below i.

Next suppose that $\pi^{-1}(i+1) > \pi^{-1}(i)$. Similarly as before we see that at any time a point $(\cdot, i+1)$ at height $i+1$ is to the right of points $(\cdot, i)$ at height i. So consider the phase when entry i is added to the tableaux. In this phase there is either no point at height $i+1$ (so it was added to the tableaux before) or the shadow line through that point needs to leave as well. Hence $i+1$ is either strictly above i or in the same row as i. □