Solutions to problem sheet 1

Problem 1.
Let \( f(n) \) denote the largest number of edges among all triangle-free graphs that are non-bipartite. For all \( n \geq 5 \) prove that

(a) \( f(n) \leq \frac{1}{4}(n - 1)^2 + 1 \),
(b) \( f(n) = \frac{1}{4}(n - 1)^2 + 1 \), if \( n \) is odd.

Solution.

(a) If \( n = 5 \) the claim is easily checked. So we assume that \( n \geq 6 \) and proceed by induction. Consider a non-bipartite, triangle-free graph \( G \) on \( n \) vertices. Then \( G \) has odd cycles since it is not bipartite. Let \( C \) be a shortest odd cycle in \( G \). Then there are no other edges in \( G[V(C)] \), since any chord in \( C \) yields a shorter odd cycle. Let \( V \) denote the set of vertices not in \( C \). Each vertex in \( V \) has at most two neighbors in \( C \) since otherwise there is a shorter odd cycle in \( G \). Indeed, if \( v \in V \) has three (or more) neighbors in \( C \) then one of the “segments” between two such neighbors has an even number of vertices, so yields an odd cycle together with \( v \). Moreover \( G[V] \) has no triangles and hence has at most \( \lfloor \frac{|V|^2}{4} \rfloor \) edges by Turán’s theorem. Since \( |V| \leq n - 5 \) the number of edges in \( G \) is at most
\[
\lfloor \frac{|V|^2}{4} \rfloor + 2|V| + n - |V| \leq \frac{(n-5)^2+4(n-5)+4(n-1)}{4} + 1 = \frac{1}{4}(n-1)^2 + 1.
\]

(b) Let \( t = \frac{n-1}{2} \). Then consider \( K_{t,t} \) minus an edge \( xy \) and add a vertex \( u \) with edges \( ux \) and \( uy \). This graph has \( n \) vertices, \( \frac{(n-1)^2}{4} + 1 \) edges, is not bipartite, and has no triangles.

Remark: Actually one can prove with the same arguments as in (a) and a construction like in (b) that \( f(n) = \text{ex}(n-1, K_3) + 1 \) for all \( n \geq 5 \). \( \square \)

Problem 2.
Consider a graph \( G \) on \( n \) vertices and \( m \) edges.

(a) Prove that \( G \) contains at least \( 4m \frac{m}{3n} (m - \frac{n^2}{4}) \) triangles.

(b) Prove that \( G \) contains at least \( \lfloor \frac{n^2}{4} \rfloor \) triangles if \( m \geq \lfloor \frac{n^2}{4} \rfloor + 1 \).

Show that the result is sharp for \( n \geq 3 \).

Solution.

(a) We follow the proof of Mantel’s theorem in Conlon’s lecture notes:

Consider a graph \( G \) on \( n \) vertices and \( m \) edges. Two adjacent vertices \( x \) and \( y \) have at least \( d(x) + d(y) - n \) common neighbors. Therefore any edge \( xy \) is contained in at least \( d(x) + d(y) - n \) triangles. Hence the total number of triangles in \( G \) is at least
\[
\frac{1}{3} \sum_{xy \in E(G)} (d(x)+d(y)-n) = \frac{1}{3} \sum_{x \in V(G)} d^2(x) - \frac{1}{3} nm \geq \frac{\sum_{x \in V(G)} d(x)^2}{3n} - \frac{1}{3} nm = \frac{4m^2}{3n} - \frac{1}{3} nm.
\]

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(b) Consider a graph \( G \) on \( n \) vertices and at least \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \) edges. For \( n \leq 2 \) there is no graph on \( n \) vertices and \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \) edges and for \( n = 3, 4 \) it is easy to see that any graph on \( n \) vertices and \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \) edges has at least \( \left\lfloor \frac{n}{2} \right\rfloor \) triangles. So we assume that \( n \geq 5 \) and proceed by induction. We distinguish whether \( n \) is odd or even.

Consider the case \( n \) is odd. Not all vertices in \( G \) are of degree at least \( \left\lfloor \frac{n}{2} \right\rfloor \), since otherwise the number of edges is at least
\[
\frac{n}{2} \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor > \left\lfloor \frac{n^2}{4} \right\rfloor + 1.
\]

Let \( v \) be a vertex of degree at most \( \left\lfloor \frac{n}{2} \right\rfloor \) and let \( G' = G - v \) be the graph obtained by removing \( v \). Then the number of edges in \( G' \) is at least
\[
\left\lfloor \frac{n^2}{4} \right\rfloor + 1 - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + 1 - \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{n-1}{2} \right\rfloor + 1.
\]

Therefore \( G' \), and hence \( G \), contains at least \( \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \) triangles and we are done.

Consider the case \( n \) is even. Assume that there is a vertex \( v \) of degree at most \( \frac{n}{2} - 1 \). Let \( G' = G - v \) be the graph obtained by removing \( v \). Then the number of edges in \( G' \) is at least
\[
\frac{n^2}{4} + 1 - \left( \frac{n}{2} - 1 \right) = \left( \frac{n}{2} - 1 \right) \frac{n}{2} + 2 = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2.
\]

Therefore \( G' \) contains a triangle. Moreover the graph obtained from \( G' \) by removing an edge from some triangle has at least \( \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1 \) edges and hence at least \( \left\lfloor \frac{n-1}{2} \right\rfloor \) triangles. Thus there are at least \( \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n}{2} \) triangles in \( G \).

Next assume that all vertices are of degree at least \( \frac{n}{2} \). For the sake of contradiction assume that \( G \) contains less than \( \left\lfloor \frac{n}{2} \right\rfloor \) triangles. Then there is an edge \( xy \) in \( G \) that is not contained in any triangle, since otherwise there are less than \( 3 \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor \) edges in \( G \). Let \( G' \) be the graph obtained from \( G \) by removing \( x \) and \( y \). The total number of edges in \( G \) incident to \( x \) or \( y \) is at most \( n - 1 \), since \( x \) and \( y \) have no common neighbors. Thus \( G' \) has at least \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 - (n - 1) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1 \) edges. Hence \( G' \) contains at least \( \left\lfloor \frac{n-2}{2} \right\rfloor \) triangles by induction. Observe that each vertex in \( G' \) is adjacent to either \( x \) or \( y \) in \( G \), since each vertex has degree at least \( \frac{n}{2} \) in \( G \). There are at most \( \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor \) edges between the neighborhoods of \( x \) and \( y \). Therefore there is an edge within one of these neighborhoods. This edge is part of a triangle in \( G \) that is not in \( G' \). Hence \( G \) contains at least \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor \) triangles.

\[\square\]

**Problem 3.**

Let \( S \) be a set of \( n \) points in \( \mathbb{R}^2 \) such that the distance between any pair of points is at most \( 1 \). Prove that there are at most \( \left\lfloor \frac{n^2}{4} \right\rfloor \) pairs of points in \( S \) whose distance is greater than \( \frac{1}{\sqrt{2}} \).

**Solution.**

Construct a graph \( G \) with the points as vertices by putting an edge \( uv \) in \( G \) if and only if the distance between \( u \) and \( v \) is greater than \( \frac{1}{\sqrt{2}} \). Then there is no \( K_4 \) in \( G \). Otherwise, in the convex hull spanned by 4 points that induce a \( K_4 \) there will be an angle of at least 90 degrees at some point \( v \) between edges \( uv \) and \( vw \). Then the distance between \( u \) and \( w \) is greater than 1, a contradiction. Therefore \( G \) has at most \( \left\lfloor \frac{n^2}{4} \right\rfloor \) edges. \[\square\]
Problem 4.
Consider a graph $H$. A graph $G$ with $|V(G)| \geq |V(H)|$ is called $H$-saturated if $G$ contains no copy of $H$, but adding any edge to $G$ yields a graph containing $H$.

For any $n \geq t \geq 3$ determine the minimum number of edges in a $K_t$-saturated graph on $n$ vertices.

Solution.
We will prove a stronger statement. A graph $G$ on $n \geq t$ vertices is called strongly $K_t$-saturated if adding any edge to $G$ yields a graph containing at least one more copy of $K_t$ than $G$. For $n \geq t \geq 3$ let $s(n,t)$ denote the minimum number of edges in a $K_t$-saturated graph on $n$ vertices and let $s^*(n,t)$ denote the minimum number of edges in a strongly $K_t$-saturated graph on $n$ vertices.

Let $G = G(n,t)$ be the graph obtained from $K_{t-2}$ and a disjoint set $S$ of $n-t+2$ isolated vertices by adding all edges between this $K_{t-2}$ and $S$. Then $G$ is $K_t$-saturated, since the largest clique in $G$ has $t-1$ vertices and adding any edges within $S$ yields a copy of $K_t$. Therefore

$$s^*(n,t) \leq s(n,t) \leq |E(G)| = \left(\frac{t-2}{2}\right) + (t-2)(n-t+2). \quad (1)$$

We will prove that any graph that has the minimum number of edges among all strongly $K_t$-saturated graphs on $n$ vertices is isomorphic to $G$. This shows that

$$s^*(n,t) = s(n,t) = \left(\frac{t-2}{2}\right) + (t-2)(n-t+2).$$

Consider $t = 3$. In this case $G$ is a star with $n$ edges. Let $H$ be a strongly $K_3$-saturated graph on $n$ vertices and $s^*(n,t)$ edges. Then $H$ is a tree, since $s^*(n,3) \leq n-1$ by inequality (1) and $H$ is clearly connected. This shows already that $s^*(n,3) = s(n,3) = n-1$. Moreover it is not hard to see that the only strongly-$K_3$-saturated tree is a star, because each pair of vertices is of distance at most 2.

Consider $t \geq 4$. If $n = t$, then $G = G(t,t)$ is a complete graph minus one edge and is clearly the unique strongly $K_t$-saturated graph on $t$ vertices (up to isomorphism). So consider $n > t$. Let $H$ be a strongly $K_t$-saturated graph on $n$ vertices and $s^*(n,t)$ edges. There are two non-adjacent vertices $x$ and $y$ in $H$. Let $H^*$ be the graph obtained from $H$ by removing $y$ and adding an edge $vx$ whenever $vy \in E(H)$ (i.e., contract $x$ and $y$).

Adding an edge to $H^*$ creates a new copy of $K_t$, since adding the corresponding edge to $H$ creates a new copy of $K_t$ that contains at most one of $x$ and $y$. So $H^*$ is strongly $K_t$-saturated. Since adding the edge $xy$ to $H$ creates a copy of $K_t$, there is a set $S$ of $t-2$ common neighbors of $x$ and $y$ in $H$ forming a complete graph. Therefore $H^*$ has at most $e(H) - (t-2) \leq \left(\frac{t-2}{2}\right) + (t-2)(n-t+2) - (t-2) = \left(\frac{t-2}{2}\right) + (t-2)((n-1)-t+2)$ edges. Inductively $H^*$ has $s^*(n-1,t)$ edges and therefore is isomorphic to $G(n-1,t)$. So there is a set $K$ of $t-2$ vertices of degree $n-2$ forming a clique in $H^*$ and the remaining vertices form an independent set. Since the vertices in $S$ form a clique in $H^*$, at most one vertex in $S$ is not in $K$. So there is a vertex $z \in S \cap K$. It is of degree $n-2$ in $H^*$, i.e., $z$ is adjacent to all other vertices in $H^*$. Since $z$ is a common neighbor of $x$ and $y$ in $H$ it has degree $n-1$ in $H$.

Thus removing $z$ from $H$ yields a strongly $K_{t-1}$-saturated graph $H'$ with at most $\left(\frac{t-2}{2}\right) + (t-2)(n-t+2) - (n-1) = \left(\frac{t-3}{2}\right) + (t-3)((n-1) - (t-3))$ edges. Inductively this number of edges equals $s^*(n-1,t-1)$ and hence $H'$ is isomorphic to $G(n-1,t-1)$. Thus $H$ is isomorphic to $G(n,m)$ since $z$ has degree $n-1$ in $H$, i.e., is adjacent to all other vertices in $H$. \[\square\]

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