Solutions to problem sheet 6

Problem 17.
Let $ex(n, C_{\geq k})$ denote the largest number of edges among all graphs on $n$ vertices that do not contain any cycle of length at least $k$, $k \geq 3$.

(a) Prove that for all $k \geq 2$ and $n \geq 1$

$$ex(n, C_{\geq k+1}) \leq \frac{k}{2} (n - 1).$$

You may use the following fact without proof: For each $k \geq 2$, each 2-connected graph with minimum degree $k$ contains a cycle of length at least $2k$ or a Hamiltonian cycle.

(b) For each $k \geq 2$ there are infinitely many values of $n$ with

$$ex(n, C_{\geq k+1}) = \frac{k}{2} (n - 1).$$

Solution.

(a) Consider an arbitrary graph $G$ on $n$ vertices with more than $\frac{k}{2} (n - 1)$ edges. Since 

$$\frac{k}{2} (n - 1) < |E(G)| \leq \binom{n}{2}$$

we have $k < n$. We shall show that $G$ contains a cycle of length at least $k + 1$ by induction on $n = |V(G)|$. If $n = k + 1$, then $G$ has more than $\frac{k}{2} (n - 1) = \frac{(n-1)^2}{2} = \binom{n}{2} - \frac{n-1}{2}$ edges. So there are at most $\lceil \frac{n-1}{2} \rceil - 1$ non-edges. Therefore $G$ has minimum degree at least $(n - 1) - \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil + 1 \geq \frac{n+1}{2}$. Thus $G$ is hamiltonian by Dirac’s theorem, that is, $G$ contains a cycle of length $n = k + 1$.

Consider $n > k + 1$. If $G$ is not 2-connected, let $G_1$ and $G_2$ denote subgraphs of $G$ such that $|V(G_1) \cap V(G_2)| \leq 1$ and $E(G) = E(G_1) \cup E(G_2)$. Then

$$|E(G_1)| + |E(G_2)| = |E(G)| > \frac{k}{2} (n - 1) \geq \frac{k}{2} (|V(G_1)| + |V(G_2)| - 2).$$

Therefore $|E(G_i)| > \frac{k}{2} (|V(G_i)| - 1)$ for some $i = 1, 2$, and hence $G_i$ contains a cycle of length at least $k + 1$ by induction. Clearly this cycle is contained in $G$.

If $v$ is a vertex of degree at most $\frac{k}{2}$ in $G$, then

$$|E(G - v)| > \frac{k}{2} (n - 1) - \frac{k}{2} = \frac{k}{2} (n - 2).$$

Therefore $G - v$ contains a cycle of length at least $k + 1$ by induction. Clearly this cycle is contained in $G$.

If $G$ is 2-connected and each vertex has degree greater than $\frac{k}{2}$, then $G$ contains a cycle of length more than $2 \frac{k}{2} = k$ by the given fact.

(b) For each $k \geq 2$ and infinitely many values of $n$ we construct graph that does not contains a cycle of length at least $k + 1$. Suppose that $k - 1$ divides $n - 1$. Then construct a graph from $\frac{n-1}{k-1}$ vertex disjoint copies of $K_{k-1}$ by adding a new vertex $v$ that is connected to all other vertices. So this graph consists of $\frac{n-1}{k-1} \binom{k}{2}$ edges, and contains no cycle of length at least $k + 1$. 

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Problem 18.

(a) For each $k \geq 1$ and infinitely many positive integers $n$ give a graph on $n$ vertices and $\frac{1}{2}(k-1)n$ edges that does not contain any tree on $k$ edges.

(b) Prove that there is a constant $c = c(k) \geq 0$ that depends only on $k$ such that for any tree $T$ on $k$ edges

$$\text{ex}(n, T) \geq \frac{1}{2}(k-1)n - c.$$

(c) Prove that $\text{ex}(n, P) \leq \frac{1}{2}(k-1)n$ for any path $P$ on $k$ edges and all $n \geq 1$.

Solution.

(a) For each $n$ that is divisible by $k$ consider the graph that consists of $\frac{n}{k}$ vertex disjoint copies of $K_k$. This graph has $\frac{n}{k} \binom{k}{2} = \frac{1}{2}(k-1)n$ edges and contains no tree on $k$ edges.

(b) Write $n = tk + r$ for integers $t$ and $r$, with $t = \lfloor \frac{n}{k} \rfloor \geq 0$, $0 \leq r \leq k - 1$. Consider the graph on $n$ vertices that consists of $t$ vertex disjoint copies of $K_k$ and another vertex disjoint copy of $K_r$. This graph contains no tree on $k$ edges and its number of edges is

$$t \binom{k}{2} + \binom{r}{2} = tk \frac{1}{2}(k-1) + \binom{r}{2}$$

$$= (n - r) \frac{1}{2}(k-1) + \frac{1}{2}(r-1)$$

$$= n \frac{1}{2}(k-1) - r \frac{1}{2}(k-r)$$

$$\geq n \frac{1}{2}(k-1) - \frac{k^2}{8}$$

Note that $x \frac{1}{2}(k-x)$ is maximized for $x = \frac{k}{2}$.

(c) Let $G$ be a graph on $n$ vertices that does not contain a path on $k$ edges. We will prove that $G$ has at most $\frac{1}{2}(k-1)n$ edges by induction on $n$. If $n \leq k$, then $G$ has at most $\binom{n}{2} \leq \frac{1}{2}(k-1)n$ edges. So assume that $n > k$. If $G$ is not connected, then each component on $n'$ vertices has at most $\frac{1}{2}(k-1)n'$ edges by induction. Let $n_1, \ldots, n_t$ denote the orders of the components of $G$. Then $G$ has at most

$$\sum_{i=1}^t \frac{1}{2}(k-1)n_i = \frac{1}{2}(k-1)n$$

If $G$ is connected, consider a longest path $P = v_1, \ldots, v_p$ in $G$. Then $p \leq k \leq n - 1$. In particular there is a vertex $u$ not in $P$ that is adjacent to some vertex in $P$. Hence the vertices $v_1, \ldots, v_p$ do not form a cycle of length $p$, i.e. a cycle using all vertices, since otherwise starting with $u$ one finds a longer path than $P$ in $G$.

Consider the edges incident to $v_1$ and $v_p$. If $v_1$ is adjacent to $v_i$, $2 \leq i \leq p-1$, then $v_p$ is not adjacent to $v_{i-1}$, since otherwise $v_1, \ldots, v_{i-1}, v_p, v_{p-1}, \ldots, v_1$ is a cycle on $V(P)$ using all vertices. This shows that the total number of edges incident to $v_1$ or $v_p$ is at most $\frac{k-1}{2}$ edges. Since the graph $G - v_1$ has at most $\frac{1}{2}(k-1)(n-1)$ edges by induction, $G$ has at most $\frac{k-1}{2}n$ edges.

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Problem 19.
Let $G$, $H$ be bipartite graphs where $G$ has parts $A$ and $B$. We call $G$ $H$-saturated, if $G$ contains no copy of $H$ but adding any edge between $A$ and $B$ to $G$ yields a graph containing $H$.
Let $G$ be an arbitrary bipartite graph with parts $A$, $|A| = m \geq s$, and $B$, $|B| = n \geq t$.

(a) Call a non-edge $e$ of $G$ between $A$ and $B$ critical, if adding $e$ to $G$ yields a bipartite graph with more copies of $K_{2,t}$ than $G$.

Prove that at most $(m-1)(n-t+1)$ of the non-edges of $G$ are critical.

(b) Prove that any $K_{2,t}$-saturated graph $G$ has at least $n + (m-1)(t-1)$ edges.

(c) Find a $K_{s,t}$-saturated graph $G$ with $(s-1)n + (m-s+1)(t-1)$ edges.

Solution.

(a) Consider an arbitrary bipartite graph $G$ with parts $A$, $|A| = m \geq 2$, and $B$, $|B| = n \geq t$. Let $ec(G)$ denote the number of critical non-edges between $A$ and $B$ in $G$. We will prove that

$$ec(G) \leq (m-1)(n-t+1).$$

Consider a critical non-edge $uw$, $u \in A$, and let $v \in A$ denote the vertex such that $\{u,v\}$ forms the part of size 2 in a new copy of $K_{2,t}$ created by adding $uw$ to $G$. Let $uw$ assign a weight of $m-1$ to $u$ and a weight of 1 to $v$. We will prove that all critical edges together assign at most $(m-1)(n-t+1)$ weight to each vertex in $A$.

Since each critical edge assigns a weight of $m$ in total this will show that

$$m \cdot ec(G) \leq |A|(m-1)(n-t+1),$$

and therefore

$$ec(G) \leq (m-1)(n-t+1).$$

Consider a fixed vertex $u \in A$. If $d_G(u) < t-1$, then $u$ is not contained in any $K_{2,t}$, even if a critical edge is added to $G$. Hence no weight is assigned to $u$ in this case. Suppose that $d_G(u) \geq t-1$. There are at most $|B| - d_G(u)$ critical edges incident to $u$ and each creates at most $m-1$ copies of $K_{2,t}$, one for each vertex in $A \setminus \{u\}$. Therefore the total weight assigned to $u$ by critical edges incident to $u$ is at most

$$(|B| - d_G(u))|A \setminus \{u\}| = (n-d_G(u))(m-1).$$

Let $E' = E'(u)$ denote the set of critical edges assigning weight 1 to $u$, i.e., edges $vw$, $v \in A$, $v \neq u$, such that $\{u,v\}$ forms the part of size 2 in a new copy of $K_{2,t}$ created by adding $vw$ to $G$. Observe that each vertex in $B$ that is incident to an edge in $E'$ is a neighbor of $u$. Moreover, if $v \in A \setminus \{u\}$ is incident to an edge in $E'$, then $v$ has at least $t-1$ common neighbors with $u$ in $G$. Therefore

$$|E'| \leq |A \setminus \{u\}|(d_G(u) - (t-1)) = (m-1)(d_G(u) - t + 1).$$

Thus the total weight assigned to a vertex in $A$ is at most

$$(m-1)(d_G(u) - t + 1) + (n-d_G(u))(m-1) = (m-1)(n-t+1).$$
(b) If $G$ is $K_{2,t}$-saturated, then each non-edge of $G$ between $A$ and $B$ is critical. Hence by part (a) the number of edges in such a graph is at least

$$ nm - (m - 1)(n - t + 1) = n + (m - 1)(t - 1). $$

(c) Consider a bipartite graph $G$ with $s-1$ vertices in part $A$ of degree $n$ and the other vertices in $A$ of degree $t-1$. Clearly, adding any edge to $G$ yields a copy of $K_{s,t}$. Therefore $G$ is $K_{s,t}$-saturated and has $(s-1)n + (m - s + 1)(t - 1)$ edges.

Similarly one can take a graph with $t-1$ vertices in $B$ degree $m$ and the other vertices in $B$ degree $s-1$. 

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