Ramsey numbers for triples

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Let \( r_3(s, t) \) be the smallest integer \( n \) such that in any red/blue coloring of the triples of an \( n \)-element set, there is either an \( s \)-element set all whose triples are red or a \( t \)-element set all whose triples are blue.

**Theorem 0.1.** If \( s \geq 3 \) and \( t \geq 4 \), then \( r_3(s, t) \leq 2^{(r_{s-1,t-1})}, r_3(3,3) = 3. \)

**Proof.** Let \( X \) be an ordered set of vertices. We write \( X' < X'' \) for subsets \( X' \) and \( X'' \) of \( X \) if each element of \( X' \) is less than every element of \( X'' \). Let \( c : \binom{X}{3} \rightarrow \{r,b\} \) be a coloring. We say that a set \( S \) of vertices makes an edge \( xy \) happy if \( \{ x, y \} \subset S \) and for any \( z \in S \) all triples \( xyz \) are of the same color. We say that a set \( S \) of vertices makes a set \( X' \) happy if \( X' < S \) and \( c(xyz) = c(xyz') \), for any \( x, y \in X', z, z' \in S \cup X' \), \( \{ x, y \} \subset \{ z, z' \} \).

Note that for any edge \( xy \) and any set \( S' > \{ x, y \} \) there is \( S \subseteq S' \), such that \( |S| \geq |S'|/2 \) and \( S \) makes \( xy \) happy. Indeed, just take the majority color class.

**Claim:** For any \( k \) s.t. \( n \geq 2^{(k)^2} \), there are sets \( X_k \) and \( S_k \) such that \( |X_k| = k, S_k \) makes \( X_k \) happy, and

\[
|S_k| \geq \frac{n}{2^{(k)^2}} - 1.
\]

Proof by induction on \( k \). When \( k = 2 \), then there is such an \( S_2 \) of size at least \( (n-2)/2 = n/2 - 1 \). Assume that the statement is true for \( k = q + 1 \). By induction there is a set \( X_q \) and a set \( S_q \) of elements greater than any element in \( X_q \) with \( |S_q| \geq n/2^{(q)^2} - 1 \geq 3 \) such that \( S_q \) makes all pairs from \( X_q \) happy and \( |X_q| = q \). Let \( x \in S_q \), be the smallest element and let \( X_{q+1} = X_q \cup \{ x \} \). We need to find a subset of \( S_q \) that makes all edges \( xy \) happy, where \( y \in X_q \). Let \( X_q = y_1, \ldots, y_q \). There is a subset \( S' \) of size \((|S_q| -1)/2\) that makes \( xy_1 \) happy, there is a subset \( S'' \) of \( S' \) that makes all \( xy_1 \) and \( xy_2 \) happy and \( |S''| \geq |S'|^2/2 \), etc., so there is a subset \( S^{(k)} \) of size \((|S_q| -1)/2^k\) that makes all \( xy \) happy, \( y \in X_q \). Let \( S_{q+1} \) be this subset \( S^{(k)} \). So, by induction

\[
|S_{q+1}| \geq (|S_q| -1)/2^q \geq \frac{n}{2^{(q)^2}} - 2 \cdot 2^{-q} \geq \frac{n}{2^{(q)^2}} - 1.
\]

This proves the Claim.

If \( s = t = 3 \) then we need to force just a single red triple or a single blue triple. So, it is clear that \( r_3(3, 3) = 3 \).

Let \( s \geq 3 \) and \( t \geq 4 \). Assume that \( n \geq 2^{(r_{s-1,t-1})} \). Let \( m = r(s-1, t-1) - 1 \). We see that \( n \geq 2^{(m+1)} \geq 2^{(3)^2} \).

By the Claim there are sets \( X_m \) and \( S_m \), such that \( S_m \) that makes \( X_m \) happy, \( |X_m| = m \) and

\[
|S_m| \geq \frac{n}{2^{(m)^2}} - 1 \geq 2^{(r_{s-1,t-1})} - (r_{s-1,t-1}) - 1 = 2^{r(s-1, t-1)} \geq 2.
\]

Let \( z \) be the smallest element in \( S_m \) and \( z' \) be any other element of \( S_m \). Then \( \{ z' \} \) makes \( X_m \cup \{ z \} \) happy.

Note that \( |X_m \cup \{ z \}| = m + 1 = r(s-1, t-1) \). Let’s color an edge \( xy \) in \( X_m \cup \{ z \} \) red if all triangles \( xyz' \) are red. Color \( xy \) blue otherwise. Then by Ramsey theorem for graphs there is a red \( K_{s-1} \) or blue \( K_{t-1} \) with vertices in \( X_m \cup \{ z \} \). Then \( X_m \cup \{ z, z' \} \), contains a set on \( s \) elements with all red triples or a set on \( t \) elements with all blue triples. 

\( \square \)