Problem 5. 5 points
Let $G$ be a non-empty graph with minimum degree at least two. Show that there is a connected graph having the same degree sequence as $G$.

Solution.
Let $G = (V,E)$ be any non-empty graph. We show the claim by induction on the number of connected components of $G$. If this is one, the graph is already connected, so assume we have two components $C_1$ and $C_2$ of $G$. By assumption of the claim, there are no vertices of degree one. So $C_i$ cannot be a tree, $i = 1, 2$. (As every tree has a leaf, as shown in the lecture.) For $i = 1, 2$, because $C_i$ is connected, it must contain a cycle and an edge $e_i = v_iw_i \in C_i$ on the cycle.

The graphs $C_i - e_i$ ($i = 1, 2$) are still connected, so $C' := (C_1 - e_1) \cup (C_2 - e_2) + v_1v_2 + w_1w_2$ is connected. Since we only touched edges incident to $\{v_1, v_2, w_1, w_2\}$ and for each of these vertices, we have removed one edge and added one edge, the degree of each vertex is preserved. The graph $(G - C_1 - C_2) \cup C'$ thus has fewer components than $G$ but same degree sequence as $G$. Applying induction to $G'$ we obtain a connected graph with the same degree sequence as $G'$ (and hence as $G$), as desired.

Problem 6. 5 points
Let $T$ be a non-empty tree with an even number of vertices. Show that $T$ has exactly one spanning subgraph in which every vertex has odd degree.

Solution. (Variant 1)
Let $T$ be a non-empty tree with an even number of vertices. We prove the claim by induction on number of vertices of $T$. First note that the desired spanning subgraph must contain all edges incident to leaves.

Now if $|V| = 2$ (recall that $T$ has an even number of vertices.) then the desired subgraph is the graph $T$ itself.

So assume that $|V| \geq 4$. Consider a vertex $x$ in $T$ such that all but at most one of its neighbors are leaves in $T$. (For example the second-to-last vertex on any maximum path in $T$ is such a vertex.) Let $x_1, \ldots, x_k$ be the leaves at $x$. Let $x'$ be the neighbor of $x$ such that $x' \neq x_i$, $i = 1, \ldots, k$. If such $x'$ does not exist, it means that $T$ is a star with odd number of edges and we are done by taking $T$ itself as the desired subgraph. Next we distinguish two cases.

---

Prof. Maria Axenovich  
http://www.math.kit.edu/iag6/edu/graphtheo2013w/
Case 1: If $k$ is odd then our spanning subgraph must contain the edge $xx_i$ for all $i = 1, \ldots, k$ and must not contain the edge $xx'$. Let $T' = T - \{x, x_1, \ldots, x_k\}$. The graph $T'$ is a tree (proved in the lecture) with even order and it has smaller number of vertices than $T$. Applying induction to $T'$ we obtain a unique spanning subgraph $G'$ with all vertices of odd order. Together with a star on vertices $x, x_1, \ldots, x_k$ we have a unique spanning subgraph of $T$ with all vertices of odd order.

Case 2: Let $k$ be even. Again, $xx_i$ must be in the desired spanning subgraph. Let $T' = T - \{x_1, \ldots, x_k\}$. The graph $T'$ is a tree of even order and $x$ is a leaf of $T'$. Thus, if $G'$ is a unique spanning subgraph of $T'$ found by induction, then $xx'$ is its edge. Therefore $G' \cup \{xx_1, \ldots, xx_k\}$ is a desired spanning subgraph of $T$. □

Solution. (Variant 2)
Let $T$ be a non-empty tree with an even number of vertices. We shall find a spanning subgraph $G$ of $T$ in which every vertex has odd degree as follows. The vertex set of $G$ is the same as of $T$, $E(G) := \{e \in E(T) \mid T_e$ has two odd components}$. First we shall prove that the degree of each vertex in $G$ is odd, second we shall prove that any spanning subgraph $G'$ of $T$ with each vertex having odd degree satisfies the property that $G' = G$.

Consider any vertex $x$ in $G$. Assume for the sake of contradiction that $d(x)$ is even. Then, deleting all edges of $G$ incident to $x$ from the tree $T$ results in odd number of odd components. Therefore, the total number of vertices in $T$ is odd – a contradiction. Thus every vertex in $G$ has odd degree.

On the other hand, consider a subgraph $G'$ of $T$ with all vertices of odd degree. Let $e$ be an edge of $G$, i.e., $T - e$ has two odd components, $T_1, T_2$. Assume for the sake of contradiction that $e \notin E(G')$. Then $G'[V(T_1)]$ has odd number of vertices of odd degree – a contradiction. Thus, if $e \in E(G)$ then $e \in E(G')$.

Now assume for the sake of contradiction that $E(G') \setminus E(G)$ contains an edge $e$. Consider the two components $T_1, T_2$ of $Te$. Both of them have even number of vertices. If $e \in E(G')$ then in $G'[V(T_1)]$ we have all vertices except for an endpoint of $e$ of odd degree, i.e., we have odd number of vertices of odd degree – a contradiction. Thus, if $e \in E(T) \setminus E(G)$ then $e \notin E(G')$.

Therefore, we have that $e \in E(G)$ if and only if $e \in E(G')$. □

Problem 7. 5 points
For any graph $G$ let $\pi(G)$ denote the minimum number of walks in $G$ so that every edge of $G$ appears once in exactly one walk and does not appear in other walks.

Find an expression/formula for $\pi(G)$.

Solution.
Let $G = (V, E)$ be any graph and $\pi(G)$ be defined as above. We define $s(G)$ to be the number of vertices in $G$ of odd degree and $t(G)$ to be the number of non-empty Eulerian components of $G$. We claim the following.

Claim. For every graph $G$ we have $\pi(G) = s(G)/2 + t(G)$.

We shall prove the claim in two steps. First, we identify a set of $s(G)/2 + t(G)$ walks in $G$ such that every edge appears exactly once in these walks, which proves $\pi(G) \leq s(G)/2 + t(G)$. And second, we prove $\pi(G) \geq s(G)/2 + t(G)$ by counting the endpoints of an arbitrary set of walks in which every edge appears exactly once in these walks.

1) We shall show that $\pi(G) \leq s(G)/2 + t(G)$. More precisely, we shall find a set $S$ of walks, $|S| = s(G)/2 + t(G)$, such that every edge of $G$ appears exactly once in
a walk in $S$. We define these walks for each connected component of $G$ separately. So let $C$ be an arbitrary non-empty connected component of $G$. We distinguish two cases.

If $C$ is Eulerian then we add any Eulerian tour (seen as a closed walk) into the set $S$. By definition every edge in $C$ appears exactly once in this walk.

On the other hand, if $C$ is not Eulerian, then it was proven in the lecture that $C$ must contain at least one vertex of odd degree. Indeed, as the sum of degrees of vertices in $G$ is even, there is an even number of vertices in $C$ that have odd degree in $G$. Let $U$ be the set of those vertices. We have $|U| = 2k$ for some natural number $k \geq 1$. We partition the vertices in $U$ into $k$ pairs of vertices, denoted by $\{u_i, w_i\}$ for $i = 1, \ldots, k$. We define a new graph $G'$ by taking only the component $C$ of $G$ and for each $i = 1, \ldots, k$ introducing a new vertex $v_i$ with edges only to $u_i$ and $w_i$.

It is easily seen that every vertex in $G'$ has an even degree. Indeed, the degree of $u_i$ and $w_i$ in $G'$ is exactly one more than its degree in $G$, the degree of a vertex in $C \setminus U$ is the same in $G$ and $G'$, and the degree of $v_i$ is two, $i = 1, \ldots, k$. Thus (as proven in the lecture) there exists a Eulerian tour of $G'$. Removing all subsequences $u_i - v_i - w_i$ for $i = 1, \ldots, k$, we obtain a set of $k$ walks in $G$ such that every edge of $G$ appears exactly once in these walks.

Altogether we have identified one walk per Eulerian component of $G$ ($t(G)$ in total) and $s(C)/2$ walks per non-Eulerian component $C$ of $G$ ($s(G)/2$ in total). Moreover, every edge appears exactly once in these walks, proving $\pi(G) \leq s(G)/2 + t(G)$.

II) We shall show that $\pi(G) \geq s(G)/2 + t(G)$. To this end consider any set $S$ of walks in $G$ such that every edge of $G$ appears exactly once in these walks. Let $C$ be an arbitrary connected component of $G$.

If $C$ is Eulerian and non-empty, then clearly at least one walk has an endpoint in $C$. And if $C$ is not Eulerian then every vertex of $C$ of odd degree is the endpoint of at least one walk in $S$. Together this shows $\pi(G) \geq s(G)/2 + t(G)$.

With I) and II) we have shown $\pi(G) = s(G)/2 + t(G)$, which concludes the proof. 

Problem 8. 5 points
A permutation matrix is a matrix of zeros and ones such that each row and each column contains exactly one 1.

Show that a square matrix $A$ with non-negative integer entries is a sum of $k$ permutation matrices if and only if the sum of elements in each row and in each column of $A$ is $k$.

Solution.
Let $A$ be any $n \times n$-matrix. First assume that $P_1, \ldots, P_k$ are $k$ permutation matrices, such that $A = P_1 + \cdots + P_k$, i.e., $A$ is the sum of $k$ permutation matrices. Then clearly the sum of elements in a row of $A$ is the sum of the corresponding row sums of $P_1, \ldots, P_k$. Since each summand is 1 (as each $P_i$ is a permutation matrix) the row sum for $A$ is exactly $k$.

Next, we show that reverse direction, that is, we assume that the sum of elements in each row and in each column of $A$ is $k$ and shall show that $A$ is the sum of $k$ permutation matrices.

We do induction on $k$. If $k = 1$, then $A$ is a permutation matrix and we are done. So let $k > 1$. We construct a bipartite graph $G$ with partite sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$.
such that $u_i v_j \in E(G)$ if and only if $A[i, j] \neq 0$. We shall show that the conditions of Halls theorem are satisfied for $G$.

Indeed, let $S = \{u_{i_1}, \ldots, u_{i_l}\}$, then $|N(S)|$ is exactly the number of columns $\{c_1, \ldots, c_q\}$ in $A$ such that some row $i_j$ has a non-zero entry in that column. Each row has total sum of elements $k$ in it. Thus total sum of elements in rows $i_1, \ldots, i_l$ is $kl$. On the other hand, if the number of columns passing through non-zero positions of these rows is less than $l$, then, since the total sum of element in these columns is at least $kl$, there is a column with at least $kl/q > kl/l = k$ sum of elements – a contradiction.

Thus Halls condition is satisfied and there is a perfect matching in $G$, say it consists of the edges $u_{i_1}v_{i_1}, \ldots, u_{i_l}v_{i_l}$. Now, the matrix $P$ such that $P[j, i_j] = 1$, $j = 1, \ldots, n$ is a permutation matrix and $A - P$ is a matrix satisfying the conditions of the problem with $k - 1$. Thus applying induction to $A - P$ we obtain $k - 1$ permutation matrices whose sum equals $A - P$, which implies that $A$ is the sum of these $k - 1$ permutation matrices and the $k$-th matrix $P$, as desired. □