A proper vertex coloring $c$ of a graph $G = (V, E)$ is called a greedy coloring if there is an order $v_1, \ldots, v_n$ of $V$ such that $c(v_1) = 1$ and for all $i = 2, \ldots, n$ we have

$$c(v_i) = \min \{ c \in \mathbb{N} \mid c(v_j) \neq c \forall v_j \in N(v_i) \text{ with } j < i \}. $$

The grundy number of $G$, denoted by $\Gamma(G)$, is the maximum $k$ for which there exists a greedy $k$-coloring of $G$.

**Problem 29.** 5 points

(a) Determine $X = \min \{ k \in \mathbb{N} \mid \Gamma(T) \leq k \text{ for all trees } T \}$.

(b) Prove for any graph $G$ that $\Gamma(G) \leq \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$.

**Solution.**

(a) We claim that $X = \infty$, namely that for every $k \in \mathbb{N}$ there exists a tree $T_k$ with $\Gamma(T_k) \geq k$. We shall prove this claim by induction on $k$.

**Induction base** $k = 1$. It clearly suffices to define $T_1$ to be the one-vertex tree.

**Induction step** $k > 1$. We construct $T_k$ as follows.

- For every $j \in [k - 1]$ take one copy of $T_j$, which exists by induction hypothesis. Let $v^1_1, \ldots, v^j_{n_j}$ be an ordering of $V(T_j)$ corresponding to a greedy coloring $c_j$ on at least $j$ colors. Moreover, let $w_j \in V(T_j)$ be a vertex with $c_j(w_j) = j$. (Indeed it holds that $w_j = v^j_{n_j}$, but we do not need it.)

- Take one extra vertex $v^*$ and define $T_k$ to be the tree composed of disjoint copies of $T_1, \ldots, T_{k-1}$ and the vertex $v^*$ with edges to $w_1, \ldots, w_{k-1}$.

We claim that $\Gamma(T_k) \geq k$. To this end consider the following vertex ordering of $T_k$

$$v^1_1, \ldots, v^1_{n_1}, v^2_1, \ldots, v^2_{n_2}, \ldots, v^{k-1}_1, \ldots, v^{k-1}_{n_{k-1}}, v^*$$

and the corresponding greedy coloring $c$. In particular, for each $i = 1, \ldots, k - 1$ the vertices of subtree $T_i$ in $T_k$ are considered in the order that forces vertex $w_i$ to get color $i$ in $c$. In the very end, $v^*$ is considered, and it has neighbors $w_1, \ldots, w_{k-1}$. Thus we obtain $c(v^*) = k$, as desired.

(b) Let $\Gamma(G) = t$ and consider a greedy $t$-coloring of $G$. By the definition of greedy colorings every vertex $x$ of color $t$ is adjacent to a vertex $y$ of color $t - 1$. Now note that $\deg(x) \geq t - 1$ and $\deg(y) \geq t - 1$. Therefore

$$\min\{\deg(x), \deg(y)\} \geq t - 1 = \Gamma(G) - 1,$$

which concludes the proof.
Now for every edge $e$

Without loss of generality let $c$ construct a proper 3-edge-coloring $c^*$ of $G^*$ from a proper 4-coloring $c$ of $G$. Solution.

Let $G$ be any plane triangulation and $G^*$ be its plane dual. First, we shall show how to construct a proper 3-edge-coloring $c^*$ of $G^*$ from a proper 4-coloring $c$ of $G$.

Show that a planar triangulation is 4-colorable if and only if its plane dual is 3-edge-colorable.

Problem 30. 5 points

Show each of the following for any graph $G$.

(a) $\text{tr}(A(G)) = 0$

(b) $\text{tr}(A(G)^2)$ equals twice the number of edges of $G$

(c) $\text{tr}(A(G)^3)$ equals six times the number of triangles in $G$

Theorem 1. For any graph $G$, $\text{tr}(A(G)) = 0$.

Proof.

□

Problem 31. 5 points

Show each of the following for any graph $G$.

(a) $\text{tr}(A(G)) = 0$

(b) $\text{tr}(A(G)^2)$ equals twice the number of edges of $G$

(c) $\text{tr}(A(G)^3)$ equals six times the number of triangles in $G$

□

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We are now ready to prove what the trace of $A$. Indeed, all three properties are implied by the following claim. Let us fix a graph $G = (V, E)$ on $n$ vertices $v_1, \ldots, v_n$ and write $A = A(G)$.

**Claim.** For all $t \geq 1$ the entry of $A^t$ at position $(i, j)$ equals the number of distinct walks in $G$ from $v_i$ to $v_j$ with exactly $t$ edges.

We prove the claim by induction on $t$. For $t = 1$ we have $A[i, j] = 1$ if $v_iv_j \in E$ and $A[i, j] = 0$ otherwise. So indeed $A[i, j]$ equals the number of distinct $v_i$-$v_j$-walks in $G$ on exactly $t = 1$ edges.

For $t \geq 2$ we have by the definition of matrix multiplication

$$A^t[i, j] = \sum_{k=1}^{n} A^{t-1}[i, k] \cdot A[k, j] = \sum_{v_k \in N(v_j)} \#	ext{walks in } G \text{ with exactly } t - 1 \text{ edges}.$$ 

As every $v_i$-$v_j$-walk in $G$ with exactly $t$ edges can be uniquely split into an edge $v_kv_j$ and an $v_i$-$v_k$-walk in $G$ with exactly $t - 1$ edges, this proves the claim.

We are now ready to prove what the trace of $A^t$ for $t = 0, 1, 2$ counts.

(a) \(\text{tr}(A) = \sum_{i=1}^{n} A[i, i] = \sum_{i=1}^{n} 0 = 0\).

(b) \(\text{tr}(A^2) = \sum_{i=1}^{n} A^2[i, i] = \sum_{i=1}^{n} \#	ext{walks with 2 edges} = \sum_{i=1}^{n} \deg(v_i) = 2|E|\).

(c) \(\text{tr}(A^3) = \sum_{i=1}^{n} A^3[i, i] = \sum_{i=1}^{n} \#	ext{walks with 3 edges, which equals 6 times the number of triangles in } G, \text{ as every closed walk on three edges is a triangle and every triangle is counted exactly 6 times.}\)

\[\square\]

**Problem 32.** 5 points

Prove that if a graph $G$ is $d$-regular, then $d$ is an eigenvalue of $A(G)$, and that if $G$ is additionally bipartite, then $-d$ is also an eigenvalue of $A(G)$.

**Solution.**

Let $G$ be an $n$-vertex graph, $A$ be its adjacency matrix, and $\lambda_{\max}$ be its largest eigenvalue. From the lecture we know that

$$\delta(G) \leq \lambda_{\max} \leq \Delta(G).$$

In particular, if $G$ is $d$-regular, then $\delta(G) = \Delta(G) = d$ and thus $\lambda_{\max} = d$.

For the second part, let us assume that $G$ is bipartite with bipartition classes $A$ and $B$. Let $w = (w_1, \ldots, w_n)$ be an eigenvector belonging to eigenvalue $\lambda_{\max} = d$. We define a vector $w' = (w'_1, \ldots, w'_n)$ by

$$w'_i = \begin{cases} w_i & \text{if } v_i \in A, \\ -w_i & \text{if } v_i \in B. \end{cases}$$

Now we compute the $i$-th coordinate of $A \cdot w'$ as

$$\sum_{j=1}^{n} A[i, j]w'_j = -\sum_{j=1}^{n} A[i, j]w_j = -\lambda_{\max}w_i = -\lambda_{\max}w'_i \quad \text{if } v_i \in A,$n$$

and

$$\sum_{j=1}^{n} A[i, j]w'_j = \sum_{j=1}^{n} A[i, j]w_j = \lambda_{\max}w_i = -\lambda_{\max}w'_i \quad \text{if } v_i \in B.$$
Thus $A \cdot w' = -\lambda_{\text{max}} w'$, i.e., $w'$ is an eigenvector of $A$ and the corresponding eigenvalue is $-\lambda_{\text{max}} = -d$. □

Open Problem.
Prove or disprove that for every connected graph $G$ and every $k \geq 1$ we have

$$t_k \geq d_k - k + 2,$$

where $d_k$ denotes the $k$-th largest degree in $G$ and $t_k$ denotes the $k$-th largest eigenvalue of $D - A(G)$, with $D[i, i] = \deg(v_i)$ and $D[i, j] = 0$ for all $i \neq j$. 

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