Solution sheet 14


Problem 53. 5 points
Show that every $\mathbb{R}$-flow $f$ on a graph $G$ decomposes into cycles, i.e., that there exists $\mathbb{R}$-flows $f_1, \ldots, f_k$ in $G$ such that $f = \sum_{i=1}^k f_i$ and for $i = 1, \ldots, k$ the set of edges with non-zero flow value in $f_i$ is a cycle.

Solution.
We proceed by induction on the number $m$ of edges with non-zero flow value in $f$.

Induction base $m = 0$. Here we have $f = 0$ and there is nothing to show.

Induction step $m \geq 1$. Let $e_1 = u_1v_1$ be any edge in $G$ with $f(u_1, v_1) \neq 0$. We define a walk in $G$ starting with $e_1$ as follows. For $i \geq 2$ let $e_i = u_iv_i$ be any edge in $G$ with $u_i = v_{i-1}$, $v_i \neq u_{i-1}$ and $f(u_i, v_i) \neq 0$. Such an edge exists as

\[ \sum_{v \in N(u_{i-1})} f(v_{i-1}, v) = 0 \]

implies that either no edge at $v_{i-1}$ carries flow, or at least two edges in $v_{i-1}$ carry flow. Now we have defined a walk in a finite graph, which thus has to contain a simple cycle $C = w_1, \ldots, w_{|C|} = w_1$. Let $f_{\text{min}} = \min\{|f(w_i, w_{i+1})| \mid i = 1, \ldots, |C| - 1\}$ and without loss of generality $w_1w_2$ be an edge with $f(w_1, w_2) = f_{\text{min}}$. We define two $\mathbb{R}$-flows $f_1$ and $f'$ by

\[ f_1(u, v) = \begin{cases} 0 & \text{if } uv \notin C \\ f_{\text{min}} & \text{if } u = w_i, v = w_{i+1} \text{ for some } i \in \{1, \ldots, |C| - 1\} \\ -f_{\text{min}} & \text{if } u = w_{i+1}, v = w_i \text{ for some } i \in \{1, \ldots, |C| - 1\}. \end{cases} \]

and

\[ f'(u, v) = f(u, v) - f_1(u, v) \quad \text{for all } uv \in E(G). \]

Now clearly, $f = f_1 + f'$ and the edges with non-zero flow value in $f'$ are a strict subset of the edges with non-zero flow value in $f$. Indeed, for edge $w_1w_2$ we have $f(w_1, w_2) \neq 0 = f'(w_1, w_2)$. In particular, $f'$ has at most $m - 1$ edges with non-zero flow. Applying induction to $f'$ we obtain $\mathbb{R}$-flows $f_2, \ldots, f_k$ each of whose edge sets with non-zero flow form a cycle such that $f' = \sum_{i=2}^k f_i$. Together we have

\[ f = f_1 + \sum_{i=2}^k f_i, \]

which concludes the proof. \qed

Problem 54. 5 points
Let $H$ be a group, $G$ be a connected graph and $T$ a spanning tree of $G$. Prove that every $H$-flow $f$ on $G$ is uniquely determined by its values on the edges not in $T$.

Solution.
Let $f$ be an $H$-flow of $G$. And $T$ be a forest in $G$. We shall show by induction on the number of edges in $T$ that $f$ is uniquely determined by its values on the edges in $G$ not in $T$.

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**Induction base** \(|E(T)| = 0\). Here is nothing to show.

**Induction base** \(|E(T)| \geq 1\). Let \(v\) be a leaf in \(T\) and \(u\) be its unique neighbor in \(T\). By the definition of an \(H\)-flow we have

\[
0 = \sum_{w \in N(v)} f(v, w) = f(v, u) + \sum_{w \in N(v), w \neq u} f(v, w),
\]

and thus \(f(v, u) = -\sum_{w \in N(v), w \neq u} f(v, w)\). We have uniquely determined \(f\) on all edges not in \(T\) and the edge \(uv\). Applying induction to \(f\) and \(T - uv\) completes the proof. \(\square\)

**Problem 55.** 5 points

Let \(T\) be the infinite complete \(k\)-ary tree with root \(r\). For some fixed \(0 < p < 1\) every edge of \(T\) is deleted independently with probability \(p\). Determine the expected number of vertices in the component containing \(r\).

**Solution.**

Let \(S\) be the set of vertices in the component containing \(r\).\(^1\) In order to calculate the expected size of \(S\) we first, we determine the probability that a given vertex \(v\) lies in \(S\).

If \(d \geq 0\) denotes the distance of \(v\) from \(r\) (counted by number of edges), then

\[
\mathbb{P}[v \in S] = (1 - p)^d
\]

since each of the \(d\) edges must no be deleted, which happens with probability \((1 - p)^d\). Now we have

\[
\mathbb{E}[|S|] = \sum_{v \in V(T)} \mathbb{P}[v \in S] = \sum_{d=0}^{\infty} k^d \cdot (1 - p)^d = \sum_{d=0}^{\infty} (k(1 - p))^d,
\]

because the number of vertices in \(T\) at distance \(d\) from the root is exactly \(k^d\). This is a geometric series of the form \(\sum_{d=0}^{\infty} a^d\) which converges if and only if \(|a| < 1\), in which case the limit is given by \(1/(1 - a)\). Thus we have

\[
\mathbb{E}[|S|] = \begin{cases} 
\infty & \text{if } p \leq \frac{k-1}{k} \\
\frac{1}{k(1-p)} & \text{if } p > \frac{k-1}{k}.
\end{cases}
\]

**Problem 56.** 5 points

A tournament is a set of \(n\) teams and one match between any two teams. Assume every match has a winner, there is no draw. Is it possible that in some tournament for every triple of teams there exists a fourth team that wins against each team in the triple?

**Solution.**

We shall prove that such tournaments indeed exist with probabilistic means. Let \(n\) be a natural number, to be determined later. Consider a random tournament \(T\) with \(n\) teams in which the winner of every match is chosen uniformly at random with probability \(1/2\).

Then for three fixed teams \(\{x, y, z\}\) and a fourth team \(w\) the probability that \(w\) wins against each of the three teams is \(2^{-3} = 1/8\). Hence, the probability that \(w\) loses against at least one of \(x, y, z\) is \(7/8\). Now we can compute the probability of the bad event as

\[
\mathbb{P}[\text{some team wins against at least one team in } \{x, y, z\}] = \left(\frac{7}{8}\right)^{n-3},
\]

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\(^1\)Bold fold indicates that this is a random variable, rather than an actual set.

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because the events for distinct $w$ are independent. We have $\binom{n}{3}$ tuples and thus by union bound we get

$$\mathbb{P}[T \text{ not good}] \leq \binom{n}{3} \left( \frac{7}{8} \right)^{n-3},$$

which is less than 1 for $n \geq 91$. Hence, as soon as $n$ is at least 91 there exists a tournament in which for every triple there is a team that wins against at least one in the triple. \qed

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**Open Problem.**
Prove or disprove that a random graph $G(2^d, \frac{1}{2})$ almost surely contains a spanning $d$-dimensional cube.