

Fork-forests in bi-colored complete bipartite graphs

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Abstract

Motivated by the problem in [6], which studies the relative efficiency of propositional proof systems, 2-edge colorings of complete bipartite graphs are investigated. It is shown that if the edges of $G = K_{n,n}$ are colored with black and white such that the number of black edges differs from the number of white edges by at most 1, then there are at least $n(1 - 1/\sqrt{2})$ vertex-disjoint forks with centers in the same partite set of G . Here, a fork is a graph formed by two adjacent edges of different colors. The bound is sharp. Moreover, an algorithm running in time $O(n^2 \log n \sqrt{n\alpha(n^2, n) \log n})$ and giving a largest such fork forest is found.

Keywords: bi-colored star forests, balanced colorings, OBDD

1 Introduction

Let $G = K_{n,n}$ with partite sets X and Y be edge colored with two colors. We investigate a global unavoidable substructure in balanced colorings of $K_{n,n}$, i.e., those where the number of edges of one color differs from the number of edges of another color by at most one. For a two-coloring c , of $E(G)$ we call a set S of three vertices a *fork* in G centered in X (or Y) if S induces two edges of different colors sharing a vertex in X (or Y). A set of vertex-disjoint forks all centered in X (or Y) is called a *fork forest* centered at X (or Y). The number of forks in a fork forest F is the *size of a forest*, denoted $|F|$. For a coloring c of G let $f(G, c)$ be the largest size of a fork forest centered either at X or at Y . Finally, let $f(n)$ be the minimum $f(G, c)$ taken over all balanced colorings c using two colors. Our main results is

Theorem 1. *For any $n > 1$, $f(n) = (1 - \frac{1}{\sqrt{2}})n$. There is an algorithm finding a largest fork forest centered at X in any two-colored complete bipartite graph with partite sets X and Y and running in time $O(n^2 \log n \sqrt{n\alpha(n^2, n) \log n})$.*

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This problem has connections to both graph theory and theoretical computer science. On one hand, it belongs to a class of problems seeking color-alternating subgraphs or general large unavoidable subgraphs in two-edge colored graphs, see for example [2, 1, 3, 5]. On the other hand, special subgraphs in bi-colored complete bipartite graphs correspond to substructures in binary matrices. Finding $f(G, c)$ allows to determine the corresponding parameter in matrices and to prove the conjecture of Tveretina et al. in [6]. In particular, our result $f(n) = (1 - \frac{1}{\sqrt{2}})n$ is an improvement of the previously known bound $f(n) \geq \frac{1}{2}(1 - \frac{1}{\sqrt{2}})n$ from [6]. This in turn improves the lower bound on resolution for ordered binary decision diagrams.

We first prove the result for $f(n)$, and then reduce the problem of finding largest fork forests to a problem of finding perfect matchings of minimum weight in edge-weighted graphs. With a known algorithm for the latter problem, our main theorem follows. In all the calculations we omit floors and ceilings when their usage is clear from the context.

2 Bounds on $f(n)$

For the upper bound, take G to be a two-colored $K_{n,n}$ with edges of one color forming a graph isomorphic to $K_{\frac{n}{\sqrt{2}}, \frac{n}{\sqrt{2}}}$.

For the lower bound consider a balanced coloring of edges of $K_{n,n}$ with partite sets X and Y in black and white. Let G_1 be the graph formed by the black edges, and let G_2 be such a graph formed by the white edges. Let M be a maximum matching of G_1 . By König's theorem applied to G_1 , there is a vertex cover S , of G_1 such that $|S| = |M|$. We have that $S \subseteq V(M)$. Let $A = V(M) \cap X$, $B = V(M) \cap Y$, $A' = A \cap S$, $B' = B \cap S$, $A'' = A - A'$, $B'' = B - B'$. Note that $|A'| = |B''|$ and $|A''| = |B'|$. Then we see that the vertex set $(X - A') \cup (Y - B')$ induces no edges in G_1 , as otherwise S would not be a vertex cover. Assume, without loss of generality, that $|A'| \geq |B'|$.

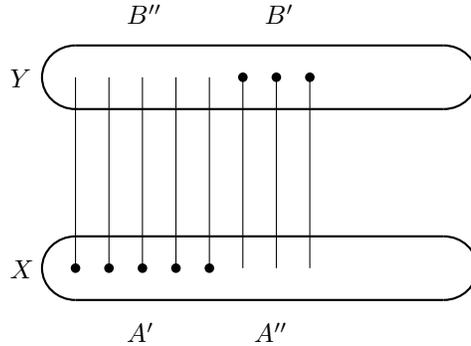


Figure 1: The matching edges M and vertex cover S of G_1 , and the labelling of vertex sets introduced above.

Case 1: $|A'| \leq \frac{n}{\sqrt{2}}$.

We have that $\frac{n^2}{2} = |E(G_1)| \leq n|A'| + (n - |A'|)|B'| \leq n|A'| + (n - |A'|)|A'| = 2n|A'| - |A'|^2$. So, from this we have that $|A'| \geq (1 - \frac{1}{\sqrt{2}})n$. Since $|X - A'| \geq (1 - \frac{1}{\sqrt{2}})n$, there is a fork forest cen-

tered at B'' , using edges of M and edges of $G_2[B'', X - A']$ with $\min\{|B''|, |X - A'|\} \geq (1 - \frac{1}{\sqrt{2}})n$ forks.

Case 2: $|A'| > \frac{n}{\sqrt{2}}$.

Let $|A'| = \frac{n}{\sqrt{2}} + c$ for some positive c . We can assume that there is a matching M' in G_2 of size at least $\frac{n}{\sqrt{2}}$, as otherwise Case 1 applies for G_2 . By counting, we can observe that at least $x := |M'| - |Y - B''| - |X - A'| \geq \frac{n}{\sqrt{2}} - 2(n - \frac{n}{\sqrt{2}} - c) = n(\frac{3}{\sqrt{2}} - 2) + 2c$ edges of M' have both endpoints in $A' \cup B''$. In the next two paragraphs we will show that there are at least $\frac{x}{2}$ forks between B'' and A' centered at B'' :

Consider the union $G' = (M \cup M')[A' \cup B'']$, i.e., the black and white matching edges with one endpoint in A' and the other in B'' . There are x edges on M' in this graph and each component is either an iterated even cycle or a path ending with edges of M . It is easy to see that one could choose at least $\frac{k}{2}$ forks centered at B'' from a component of G' containing k edges of M' and that is either a path or a cycle of length divisible by 4. We also observe that one can choose $\frac{1}{2}(k_1 + k_2)$ forks centered at B'' from two cycles of G' with k_1 and k_2 edges of M' , where k_1 and k_2 are odd, by using a single edge between these cycles and additional edges from the cycles.



Figure 2: Finding forks in the components of G' , with one example for each type. White edges are drawn in light gray, the vertices belonging to B'' are positioned at the top. The outlined edges have been chosen to be used in forks. In the latter two cases this choice depends on the color of the single edge not contained in the cycles.

So, we can pair up all but at most one of the components of G' that are cycles of length 2 modulo 4. In the remaining such component with k edges of M' we can choose $\frac{k+1}{2}$ forks centered at B'' by using one additional edge going from the component into a previously unused vertex in A' if available. If not, then all vertices in A' have been used up from the previously chosen forks, so we already have got $\lfloor \frac{1}{2}|A'| \rfloor \geq \lfloor \frac{n}{2\sqrt{2}} \rfloor$ forks. By combining the selected forks, we see that there are at least $\frac{x}{2}$ forks centered in B'' and having leaves in A' .

We observe that with each chosen fork, at most two matching edges of M have become unavailable for later use, so there are at least $|B''| - x$ black matching edges with both endpoints in $A' \cup B''$ remaining. These can be combined into forks centered at B'' with nonedges leading into $X - A'$. This results in a total of $\frac{x}{2} + \min\{|B''| - x, |X - A'|\}$ forks. Since

$$\begin{aligned} \frac{x}{2} + \min\{|B''| - x, |X - A'|\} &\geq n(\frac{3}{2\sqrt{2}} - 1) + c + \min\{n(2 - \frac{2}{\sqrt{2}}) - c, n(1 - \frac{1}{\sqrt{2}}) - c\} \\ &= \min\{n(2 - \frac{2}{\sqrt{2}} + \frac{3}{2\sqrt{2}} - 1), n(1 - \frac{1}{\sqrt{2}} + \frac{3}{2\sqrt{2}} - 1)\} \\ &= \min\{n(1 - \frac{1}{2\sqrt{2}}), n(\frac{1}{2\sqrt{2}})\} = \frac{n}{2\sqrt{2}} \geq n(1 - \frac{1}{\sqrt{2}}), \end{aligned}$$

it follows that $f(G, c) \geq n(1 - \frac{1}{\sqrt{2}})$.

3 Algorithm

We show that there is an efficient algorithm for finding the largest fork forest centered at X in G by reducing this problem to the problem of finding a perfect matching of minimum weight in an edge-weighted graph G' . The case of a fork forest centered at Y is symmetric.

Informally, G' is obtained from G by first splitting each vertex of X into two adjacent vertices, with one of them being assigned the black edges incident to the original vertex, and the other taking the white edges. Then all edges in Y are added, and if n is odd, one additional vertex is added adjacent to all vertices of Y .

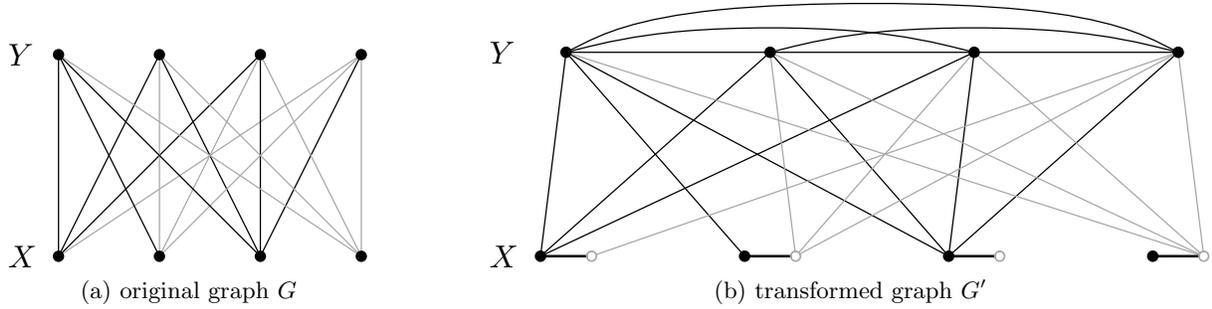


Figure 3: A coloring of $G = K_{4,4}$ with white edges drawn in light gray, and its transformed version G' on the right. For $x \in X$, vertices $x_b \in G'$ with the black edges incident to them are drawn in black, while $x_w \in G'$ with white incident edges are drawn in light gray.

Construction

For a $\{b, w\}$ -coloring, c , of $G = K_{n,n}$ with partite sets X and Y , let $V(G')$ be a disjoint union $Y' \cup \{x_b : x \in X\} \cup \{x_w : x \in X\}$, where $Y' = Y$ if n is even and $Y' = Y \cup \{y\}$ if n is odd. Let $E(G')$ be the union of $\{x_b x_w : x \in X\}$, $\{y x_b : c(yx) = b, x \in X, y \in Y\}$, $\{y x_w : c(yx) = w, x \in X, y \in Y\}$, and all possible edges with endpoints in Y' . Let $\tau : E(G') \rightarrow \{0, 1\}$ be such that $\tau(x_b x_w) = 1$ for all $x \in X$, and $\tau(e) = 0$, for all other edges.

Further, if M is a perfect matching in G' , denote by $\text{fork}(M)$ a fork forest in G containing all forks on vertices x, y, y' if $x_b y \in M, x_w y' \in M$. Recall that $|\text{fork}(M)|$ is the number of forks in $\text{fork}(M)$.

Lemma 1. *If M is a minimum weight perfect matching of (G', τ) then $\text{fork}(M)$ is a maximum fork forest of (G, c) centered at X .*

Proof. Let M be a minimum weight perfect matching of (G', τ) . Note that the weight of M is equal to the number of edges $x_b x_w \in E(M)$. We see that $x \notin V(\text{fork}(M))$ if and only if $x_b x_w \in E(M)$, so the weight of M is $n - |\text{fork}(M)|$.

Assume that $\text{fork}(M)$ is not a largest fork forest of (G, c) centered at X . Then, for a larger fork forest F' of (G, c) centered at X , let M' be a perfect matching of G' that contains edges x_by and x_wy' if x, y, y' induces a fork of F' , and edge x_bx_w , otherwise. Note that one can always match vertices of Y that are not in F' with remaining vertices of Y' . This matching M' has weight $n - |F'| < n - |\text{fork}(M)|$, a contradiction. \square

In [4] it is shown that the time complexity of finding the minimum weight matching in a graph with n vertices, m edges, and edge-weights 0 or 1 is $O(\sqrt{n\alpha(m, n)} \log nm \log n)$, where α denotes the slowly growing inverse of the Ackermann function. Since G' contains at most $3n + 1$ vertices and $\frac{3}{2}(n^2 + n)$ edges, the minimum weight perfect matching problem for (G', w) can be solved in $O(n^2 \log n \sqrt{n\alpha(n^2, n)} \log n)$ time. Thus, the main theorem follows.

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