

# DENSITY OF RANGE CAPTURING HYPERGRAPHS

MARIA AXENOVICH AND TORSTEN UECKERDT

ABSTRACT. For a finite set  $X$  of points in the plane, a set  $S$  in the plane, and a positive integer  $k$ , we say that a  $k$ -element subset  $Y$  of  $X$  is captured by  $S$  if there is a homothetic copy  $S'$  of  $S$  such that  $X \cap S' = Y$ , i.e.,  $S'$  contains exactly  $k$  elements from  $X$ . A  $k$ -uniform  $S$ -capturing hypergraph  $\mathcal{H} = \mathcal{H}(X, S, k)$  has a vertex set  $X$  and a hyperedge set consisting of all  $k$ -element subsets of  $X$  captured by  $S$ . In case when  $k = 2$  and  $S$  is convex these graphs are planar graphs, known as *convex distance function Delaunay graphs*.

In this paper we prove that for any  $k \geq 2$ , any  $X$ , and any convex compact set  $S$ , the number of hyperedges in  $\mathcal{H}(X, S, k)$  is at most  $(2k - 1)|X| + O(k^2)$ . Moreover, this bound is tight up to an additive  $O(k^2)$  term. This refines a general result of Buzaglo, Pinchasi and Rote [2] stating that every pseudodisc topological hypergraph with vertex set  $X$  has  $O(k^2|X|)$  hyperedges of size  $k$  or less.

*Keywords:* Hypergraph density, geometric hypergraph, range-capturing hypergraph, homothets, convex distance function, Delaunay graph.

## 1. INTRODUCTION

Let  $S$  and  $X$  be two subsets of the Euclidean plane  $\mathbb{R}^2$  and  $k$  be a positive integer. In this paper,  $S$  is a convex compact set containing the origin and  $X$  is a finite set. We call a homothetic copy of  $S$ , an  *$S$ -range* or simply range, when it is clear from the context. In other words, a range is obtained from  $S$  by first contraction or dilation and then translation. Two  $S$ -ranges are *contractions/dilations* of one another if in their corresponding mappings the origin is mapped to the same point.

We say that an  $S$ -range  $S'$  *captures* a subset  $Y$  of  $X$  if  $X \cap S' = Y$ . An  *$S$ -capturing hypergraph* is a hypergraph  $\mathcal{H} = (X, \mathcal{E})$  with vertex set  $X$  and edge set  $\mathcal{E} \subseteq 2^X$  such that for every  $E \in \mathcal{E}$  there is an  $S$ -range  $S'$  that captures  $E$ .

In this paper we consider  *$k$ -uniform  $S$ -capturing hypergraphs*, that is, those hypergraphs  $\mathcal{H} = \mathcal{H}(X, S, k)$  with vertex set  $X$  and hyperedge set consisting of *all*  $k$ -element subsets of  $X$  captured by  $S$ . I.e., the hyperedges correspond to ranges containing exactly  $k$  elements from  $X$ . These hypergraphs are often referred to as *range-capturing hypergraphs* or *range spaces*. The importance of studying  $k$ -uniform  $S$ -capturing hypergraphs was emphasized by their connection to epsilon nets and covering problems of the plane [10, 11, 14]. See also some related literature for geometric hypergraphs, [1–4, 6–9, 12, 13, 15–17, 21, 22, 24].

The first non-trivial case  $k = 2$ , i.e., when  $\mathcal{H}(X, S, k)$  is an ordinary graph, was first considered by Chew and Dyrsdale in 1985 [5]. They show that if  $S$  is convex and compact, then  $\mathcal{H}(X, S, 2)$  is a planar graph, called the *Delaunay graph* of  $X$  for the *convex distance function* defined by  $S$ . In particular,  $\mathcal{H}(X, S, 2)$  has at most  $3n - 6$  edges and this bound can be achieved. Indeed, it follows from Schnyder's realizer [19] that every maximally planar graph can be written as  $\mathcal{H}(X, S, 2)$  for some  $X$  and  $S$  being any triangle.

### 1.1. Related work.

Recently, Buzaglo, Pinchasi and Rote [2] considered the maximum number of hyperedges of size  $k$  or less in a pseudodisc topological hypergraph on  $n$  vertices. Here, a *family of pseudodiscs* is a set of closed Jordan curves such that any two of these curves either do not intersect or intersect in exactly two points. A hypergraph is called *pseudodisc topological hypergraph* if its vertex set  $X$  is a set of points in the plane and for every hyperedge  $Y$  there is a closed Jordan curve such that the bounded region of the plane obtained by deleting the curve contains  $Y$  and no point from  $X \setminus Y$ , and the set of all these Jordan curves is a family of pseudodiscs.

The authors of [2] observe that pseudodisc topological hypergraphs have VC-dimension [23] at most 3, and that using this fact the number of hyperedges can be bounded from above. For this, a version of the Perles-Sauer-Shelah theorem [18, 20] is applied. Let, for a set  $A$  and a positive integer  $d$ ,  $\binom{A}{\leq d}$  denote the set of all subsets of  $A$  of size at most  $d$ .

**Theorem 1** (Perles-Sauer-Shelah Theorem). *Let  $F = \{A_1, \dots, A_m\}$  be a family of distinct subsets of  $\{1, 2, \dots, n\}$  and let  $F$  have VC-dimension at most  $d$ . Then  $m \leq \left| \bigcup_{i=1}^m \binom{A_i}{\leq d} \right|$ .*

Applying this theorem to the family of hyperedges in a pseudodisc topological hypergraph, one can see that the number of hyperedges in such a hypergraph is at most  $O(n^3)$ . In fact, if one considers only hyperedges of size  $k$  or less, a much stronger bound could be obtained.

**Theorem 2** (Buzaglo, Pinchasi and Rote [2]). *Every pseudodisc topological hypergraph on  $n$  vertices has  $O(k^2n)$  hyperedges of size  $k$  or less.*

However, the methods used to prove Theorem 2 do not seem to give any non-trivial bound on the number of hyperedges of size *exactly*  $k$ .

### 1.2. Our results.

In this paper, we consider the case when every hyperedge has **exactly  $k$  points**. In particular, we consider  $k$ -uniform  $S$ -capturing hypergraphs, for convex and compact sets  $S$ . One can show that these hypergraphs are pseudodisc topological hypergraphs. Indeed, the family of all homothetic copies of a fixed convex set  $S$  is surely the most important example of a family of pseudodiscs.

**Theorem 3** (Upper Bound). *Let  $S$  be a convex compact set and  $k, n$  be positive integers. Any  $k$ -uniform  $S$ -capturing hypergraph on  $n$  vertices has at most  $(2k - 1)n + O(k^2)$  hyperedges.*

Note that for  $k = 2$  this amounts for at most  $3n + c$  edges for some absolute constant  $c$ . We obtain the following refinement of Theorem 2.

**Corollary 4.** *Let  $S$  be a convex compact set and  $k, n$  be positive integers. Any  $S$ -capturing hypergraph on  $n$  vertices has at most  $k^2n + O(k^3)$  hyperedges of size  $k$  or less.*

We show that we may assume without loss of generality that  $S$  has “no corners” and “no straight segments on its boundary”. We call such sets *nice shapes* and define them formally later.

**Theorem 5** (Lower Bound). *For any positive integers  $k$  and  $n$  and any nice shape  $S$ , there exists a  $k$ -uniform  $S$ -capturing hypergraph on  $n$  vertices with at least  $(2k - 1)n - O(k^2)$  hyperedges.*

In fact, Theorem 5 also holds for any convex compact and not necessarily nice shape  $S$ , as we indicate later. We prove the upper bound counting capturing ranges of two different types. In order to obtain the sharp bound, we introduce a notion of *suspension* that builds an auxiliary point set system with additional properties.

The paper is organized as follows. Section 2 provides general definitions, reduction of an arbitrary capturing hypergraph to one with a *nice* shape  $S$  and a point set in a *nice* position. This section also contains the definition of suspension. Section 3 proves Theorem 5. Section 4 introduces different types of ranges. An upper and lower bound on the number of hyperedges of Type I is proven in Section 4.1. We also give a relatively easy but not tight upper bound on the total number of hyperedges in this section. A more delicate analysis from Section 4.2 gives an inequality involving the number of ranges of both types in  $X$  and in a suspension of  $X$ . Finally, Theorem 3 is proven in Section 5.

## 2. GENERAL POSITION ASSUMPTIONS, SUSPENSION, NEXT RANGE

In this section we introduce nice shapes, the concepts of the next range and a suspended point set and state their basic properties. For the ease of reading the proofs of all the results in this section are provided in the appendix because they are quite straightforward but also technical. We denote the boundary of a set  $S$  by  $\partial S$ . We denote the line through points  $p$  and  $q$  by  $\overline{pq}$ . A halfplane defined by a line  $\ell$  is a connected component of the plane after the removal of  $\ell$ . In particular, such halfplanes are open sets. Typically we denote the two halfplanes defined by a line  $\ell$  as  $L$  and  $R$ , which can stand for “left” and “right”. However, the situation is completely symmetric and hence all statements remain true when  $L$  and  $R$  are interchanged.

### 2.1. Nice shapes and general position of a point set.

A convex compact set  $S$  is called a **nice shape** if

- (i) for each point in  $\partial S$  there is exactly one line that intersects  $S$  only in this point and
- (ii) the boundary of  $S$  contains no non-trivial straight line segment.

For example, a disc is a nice shape, but a rectangle is not. A nice shape has no “corners” and we depict nice shapes as discs in most of the illustrations.

**Lemma 6.** *If  $S$  is a nice shape,  $S_1$  and  $S_2$  are distinct  $S$ -ranges, then each of the following holds.*

- (i)  $\partial S_1 \cap \partial S_2$  is a set of at most two points.
- (ii) If  $\partial S_1 \cap \partial S_2 = \{p, q\}$  and  $L$  and  $R$  are the two open halfplanes defined by  $\overline{pq}$ , then
  - $S_1 \cap L \subset S_2 \cap L$  and  $S_1 \cap R \supset S_2 \cap R$  or
  - $S_1 \cap L \supset S_2 \cap L$  and  $S_1 \cap R \subset S_2 \cap R$ .
- (iii) Any three non-collinear points lie on the boundary of a unique  $S$ -range.
- (iv) For a subset of points  $X \subset \mathbb{R}^2$  and any  $Y \subset X$ ,  $|Y| \geq 2$ , that is captured by some  $S$ -range there exists at least one  $S$ -range  $S'$  with  $Y = X \cap S'$  and  $|\partial S' \cap X| \geq 2$ .

The proof of Lemma 6 is provided in the Appendix. We remark that only the last item of Lemma 6 remains true if  $S$  is convex compact but not nice. For example, if  $S$  is an axis-aligned square, then no three points with strictly monotone  $x$ - and  $y$ -coordinates lie on the boundary of any  $S$ -range, whereas three points, two of which have the same  $x$ - or  $y$ -coordinate lie on the boundary of infinitely many  $S$ -ranges.

For a set  $S \subset \mathbb{R}^2$  we say that  $X \subset \mathbb{R}^2$  is **in general position with respect to  $S$**  if

- (i) no three points of  $X$  are collinear,

- (ii) no four points of  $X$  lie on the boundary of any  $S$ -range,
- (iii) no two points of  $X$  are on a vertical line.

**Lemma 7.** *For any point set  $X$ , positive integer  $k$  and a convex compact set  $S$ , there is a point set  $X'$  and a nice shape  $S'$ , such that  $|X'| = |X|$  and the number of edges in  $\mathcal{H}(X', S', k)$  is at least as large as the number of edges in  $\mathcal{H}(X, S, k)$ .*

To prove Lemma 7, we show that one can move the points of  $X$  slightly and modify  $S$  slightly to obtain the desired property. See the Appendix for a detailed account of the argument. From now on we will always assume that  $S$  is a nice shape and  $X$  is a finite point set in a general position with respect to  $S$ .

## 2.2. Next Range.

Let  $p$  and  $q$  be two points in the plane,  $L$  and  $R$  be the two halfplanes defined by  $\overline{pq}$  and  $S_1$  be any  $S$ -range with  $p, q \in \partial S_1$ . Let  $S_2$  be an  $S$ -range with

$$p, q \in \partial S_2 \quad \text{and} \quad (\partial S_2 \setminus \partial S_1) \cap X \neq \emptyset \quad \text{and} \quad S_1 \cap L \subset S_2 \cap L.$$

If  $S_2 \cap L$  is inclusion minimal among all  $S$ -ranges satisfying the above properties, then we call  $S_2$  the **next range** of  $S_1$  in  $L$  and denote it  $\text{next}_L(S_1)$ . Throughout this paper we always assume that  $|\partial S_1 \cap (X \setminus \{p, q\})| \leq 1$  and no four points of  $X \cup \{p, q\}$  lie on the boundary of any  $S$ -range. This implies that if  $S_1$  captures  $k$  elements of  $X$  then  $\text{next}_L(S_1)$  captures  $k - 1$ ,  $k$  or  $k + 1$  elements of  $X$ . Indeed, the following holds.

$$\text{If } R \cap \partial S_1 \cap X \neq \emptyset, \text{ then } \text{next}_L(S_1) \text{ captures } k \text{ or } k - 1 \text{ points from } X. \quad (1)$$

$$\text{If } R \cap \partial S_1 \cap X = \emptyset, \text{ then } \text{next}_L(S_1) \text{ captures } k \text{ or } k + 1 \text{ points from } X. \quad (2)$$

See Figure 1 for the three possible case scenarios.

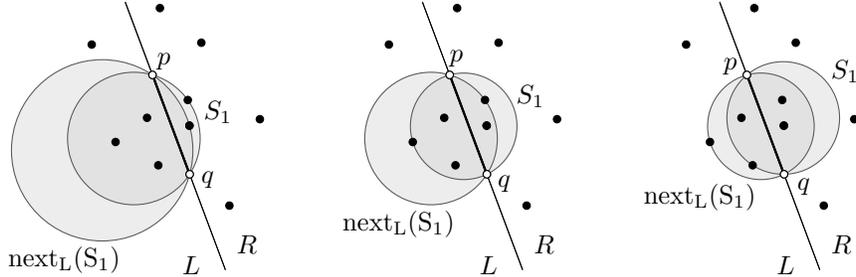


FIGURE 1. Three cases of an  $S$ -range  $S_1$  with two boundary points  $p$  and  $q$ , and the next  $S$ -range of  $S_1$  in a halfplane  $L$  defined by  $\overline{pq}$ . Note that  $|X \cap \text{next}_L(S_1)| - |X \cap S_1|$  is  $-1$  on the left,  $0$  in the middle, and  $1$  on the right.

Informally, we can imagine continuously transforming  $S_1$  into a new  $S$ -range containing  $p$  and  $q$  on its boundary and containing  $S_1 \cap L$  and all points of  $(S_1 \setminus \partial S_1) \cap X$ . As soon as this new  $S$ -range contains a point from  $X \setminus \{p, q\}$  on its boundary, we choose it as a next range of  $S_1$  in  $L$ . Note that if  $S_2 = \text{next}_L(S_1)$  and  $\partial S_1$  contains a point from  $X \setminus \{p, q\}$ , then  $\text{next}_R(S_2) = S_1$ .

Let us denote the symmetric difference of two sets  $A$  and  $B$  by  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .

**Lemma 8.** *Let  $S_1$  be any  $S$ -range,  $p, q$  be two distinct points on  $\partial S_1$ , and  $L, R$  be the two halfplanes defined by  $\overline{pq}$ . Then the following holds.*

- (i) If  $(S_1 \triangle L) \cap X \neq \emptyset$  then  $\text{next}_L(S_1)$  exists and is unique.
- (ii) If  $S_2 = \text{next}_L(S_1)$  then  $S_2 \cap R \subset S_1 \cap R$ .

The proof of Lemma 8 is provided in the Appendix.

### 2.3. Suspension.

For a set  $X$  in the plane, consider a disc  $D$  centered at the origin and containing  $X$ . Let  $D$  have diameter  $d$ . Consider an equilateral triangle  $a, b, c$  with center at the origin and side length  $5d$ . Finally consider three discs  $D_a, D_b, D_c$  centered at  $a, b, c$  respectively and with diameter  $d$  each. Let  $A \subset D_a, B \subset D_b, C \subset D_c, |A| = |B| = |C| = k$ . If  $X' = X \cup A \cup B \cup C$  is in general position with respect to  $S$ , then  $X'$  is called a  $k$ -suspension or simply **suspension** of  $X$ . See Figure 2 for an illustration.

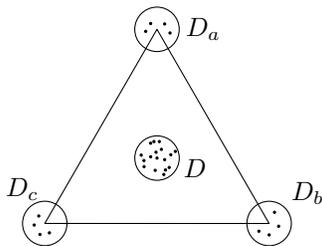


FIGURE 2. A point set  $X$  inside disc  $D$  is suspended by the addition of three sets of  $k$  points each inside discs  $D_a, D_b, D_c$ .

**Lemma 9.** *Let  $X$  be a set of points and  $X'$  be a  $k$ -suspension of  $X$ . Then for any  $p \in X$  and any  $q \in X', p \neq q$ , the line  $\overline{pq}$  separates the plane into two halfplanes each containing at least  $k$  elements of  $X'$ .*

*Proof.* It suffices to note that if a line  $\ell$  intersects  $D$  then it intersects at most one of  $D_a, D_b, D_c$ . Moreover, two of  $D_a, D_b, D_c$  lie on different sides of  $\ell$ .  $\square$

### 3. LOWER BOUND

*Proof of Theorem 5.* Let  $S$  be any nice shape. First note that when  $k = 1$ , then any set  $X$  will do. So let  $k \geq 2$  and  $n > 2(k - 1)$  be any given natural numbers. We shall define a set  $X$  of  $n$  points in the plane such that  $\mathcal{H}(X, S, k)$  has at least  $(2k - 1)n - O(k^2)$  hyperedges.

We consider three horizontal lines  $\ell_A, \ell_0, \ell_B$ , where  $\ell_A$  is above  $\ell_0$ , which is above  $\ell_B$ . We place a set  $X_0$  of  $n - 2(k - 1) > 0$  points on  $\ell_0$  and a set  $A$  and  $B$  of  $k - 1$  points each on  $\ell_A$  and  $\ell_B$ , respectively. We allow the distance between  $\ell_A$  and  $\ell_0$ , and between  $\ell_0$  and  $\ell_B$  to be large enough so that every  $j$  consecutive points in  $X_0$ , for  $1 \leq j \leq k$ , can be captured with an  $S$ -range which is disjoint from  $A \cup B$ . Setting  $X = X_0 \cup A \cup B$  completes the definition of  $X$  and hence also  $\mathcal{H}(X, S, k)$ . See Figure 3 for an illustration.

We label the points in  $X_0$  by  $p_1, \dots, p_{n-2(k-1)}$  in the increasing order of  $x$ -coordinates and consider a point  $p_i \in X_0$  with  $k \leq i \leq n - 2(k - 1)$ . We claim that there are at least  $2k - 1$  hyperedges in  $\mathcal{H}(X, S, k)$  whose rightmost point in  $X_0$  is  $p_i$ . To this end, consider for each  $j \in \{1, \dots, k\}$  the set  $Y_j$  of  $j$  consecutive points of  $X_0$  whose rightmost point is  $p_i$ . We shall show that for  $j < k$  there are 2 hyperedges of  $\mathcal{H}(X, S, k)$  that intersect  $X_0$  in  $Y_j$ , and for  $j = k$  there is 1 such hyperedge.

For  $j = 1$  we have  $Y_j = \{p_i\}$ . Consider all  $S$ -ranges that have  $\ell_0$  as a touching line at  $p_i$ . There is such a range intersecting  $X$  only by  $p_i$  and located above  $\ell_0$ . Increase this

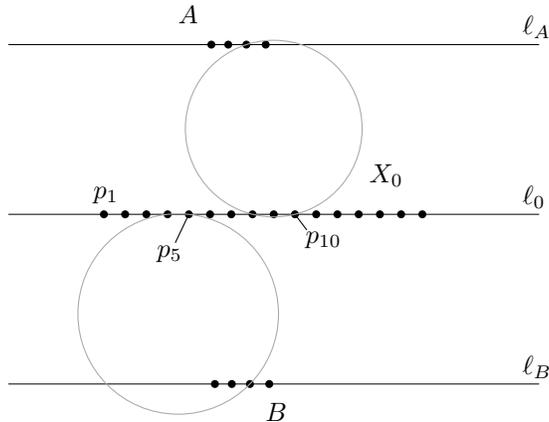


FIGURE 3. Lower bound example with a set  $X_0$  of  $n - 2(k - 1)$  points on a line  $\ell_0$  and two sets  $A$  and  $B$  of  $k - 1$  points each on lines  $\ell_A$  and  $\ell_B$  above and below  $\ell_0$ . Here  $k = 5$  and  $n = 24$ .

range continuously so that  $\ell_0$  remains a touching line. At some point an  $S$ -range will contain exactly  $k - 1$  points from  $A$  and exactly one point,  $p_i$  from  $X_0$ . Similarly, there is an  $S$ -range containing  $p_i$  from  $X_0$  and exactly  $k - 1$  points from  $B$ .

For  $2 \leq j \leq k$ , consider an  $S$ -range  $S'$  with  $\partial S' \cap X_0 = \{p_{i-j+1}, p_i\}$  and  $S' \cap (A \cup B) = \emptyset$ . Clearly,  $S' \cap X_0 = Y_j$ . If  $j = k$  we have found the desired  $S$ -range. So assume that  $j < k$ . Let  $S'_1 = \text{next}(S')$  with respect to a halfplane above  $\ell_0$  and  $\{p_{i-j+1}, p_i\}$ . Then  $S'_1$  contains exactly one point from  $A$  and  $|S'_1 \cap X| = j + 1$ . By Lemma 8 (ii)  $S'_1$  does not contain any points from  $B$ . Continue applying the *next* operation with respect to a halfplane above  $\ell_0$  and  $\{p_{i-j+1}, p_i\}$  to the obtained range, we get  $S'_{k-j}$  - a range intersecting  $X_0$  in a set  $Y_j$ , intersecting  $A$  in exactly  $k - j$  points and not intersecting  $B$ . Similarly we can find a range that intersects  $X_0$  in  $Y_j$ , intersects  $B$  in exactly  $k - j$  points, and does not intersect  $A$ . See Figure 3.

This gives in total  $1 + 2(k - 1) = 2k - 1$  hyperedges whose rightmost point in  $X_0$  is  $p_i$ . Summing over all  $i$  from  $k$  to  $n - 2(k - 1)$  we have at least  $(2k - 1)(n - 3k) = (2k - 1)n - O(k^2)$  hyperedges in  $\mathcal{H}(X, S, k)$ .  $\square$

*Remark 10.* If the set  $A$  is placed sufficiently far to the left of  $X_0$  and the set  $B$  is placed sufficiently far to the left of  $A$ , then a more careful counting shows that the hypergraph defined in the above proof has exactly  $(2k - 1)n - 2(k - 1)(2k - 1)$  hyperedges. Note that for  $k = 2$  this amounts for  $3n - 6$  edges, which is best-possible, since every hypergraph  $\mathcal{H}(X, S, 2)$  is a planar graph.

*Remark 11.* It is possible to modify the proof of Theorem 5 for convex bounded sets  $S$  that are not necessarily nice. To this end, assume without loss of generality that  $\partial S$  contains no non-trivial horizontal line segment, and that there exists an  $S$ -range capturing  $X_0 \cup A$  but no point from  $B$  and another  $S$ -range capturing  $X_0 \cup B$  but no point from  $A$ .

#### 4. REPRESENTATIVE $S$ -RANGES AND TYPES I AND II

Let  $X$  be a set in a general position and  $S$  be a nice shape. Let  $Y$  be any hyperedge in  $\mathcal{H}(X, S, k)$ . An  $S$ -range  $S'$  is a *representative  $S$ -range for  $Y$*  if  $Y = X \cap S'$  and among all such  $S$ -ranges  $S'$  has the maximum number of points from  $Y$  on its boundary. From

Lemma 6 (iv) it follows that each hyperedge has a representative range and if  $S'$  is a representative  $S$ -range for  $Y$ , then  $S'$  has two or three points of  $X$  on its boundary.

We say that  $S'$  is of **Type I** if  $|\partial S' \cap X| = 3$  and of **Type II** if  $|\partial S' \cap X| = 2$ .

We say that  $Y$  is of *Type I* if it has a representative range of Type I, otherwise it is of *Type II*. Note that in total we have at most  $\binom{k}{3}$  many Type I ranges representing a Type I hyperedge since by Lemma 6 (iii) any three points of  $X$  are on the boundary of only one  $S$ -range. On the other hand, every Type II hyperedge has infinitely many representative ranges, see Figure 4. The **representative set** of  $Y$  contains all representative ranges for a Type I set  $Y$  and it contains one arbitrarily chosen representative range for a Type II set  $Y$ . We denote a representative set of  $Y$  by  $\mathcal{R}(Y)$ .

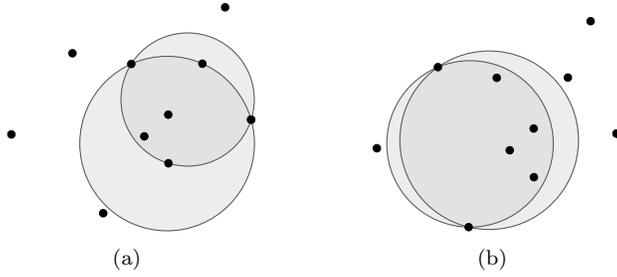


FIGURE 4. (a) A Type I hyperedge and with only two representative ranges.  
 (a) A Type II hyperedge with two of its infinitely many representative ranges.

For a hyperedge  $Y$ , we define the graph  $G(Y) = (Y, E_Y)$  with an edge set  $E_Y = \{\{p, q\} : p, q \in \partial S', S' \in \mathcal{R}(Y)\}$ . If  $Y$  is of Type I, then  $G(Y)$  is the union of triangles, one for each representative range of  $Y$ , and isolated vertices. We call an edge of  $G(Y)$  *inner edge* if it is contained in at least two triangles.

**Lemma 12.**

- (i) For a hyperedge  $Y$  of Type I,  $G(Y)$  is outerplanar and the number of inner edges of  $G(Y)$  is at most  $|\mathcal{R}(Y)| - 1$ .
- (ii) All Type II representative  $S$ -ranges for the same hyperedge have the same pair of  $X$  in their boundary.

We prove Lemma 12 in the Appendix. Note that  $|\mathcal{R}(Y)|$  is at most the number of triangles in  $G(Y)$ , which by Lemma 12 is at most  $k - 2$ .

We denote the hyperedges of Type I and II, respectively, by

$$\mathcal{E}_1^k(X) \text{ and } \mathcal{E}_2^k(X).$$

We further define

$$\mathcal{R}_1^k(X) = \{S \in \mathcal{R}(Y) : Y \in \mathcal{E}_1^k(X)\} \text{ and } \mathcal{R}_2^k(X) = \{S \in \mathcal{R}(Y), Y \in \mathcal{E}_2^k(X)\}.$$

From the definitions we have

$$|\mathcal{E}_1^k(X)| = |\mathcal{R}_1^k(X)| - \sum_{Y \in \mathcal{E}_1^k(X)} (|\mathcal{R}(Y)| - 1) \quad \text{and} \quad |\mathcal{E}_2^k(X)| = |\mathcal{R}_2^k(X)|. \quad (3)$$

#### 4.1. Upper and lower bounds on the number of Type I ranges.

**Proposition 13.**  $|\mathcal{R}_1^k(X)| \leq 2(k-2)|X|$  and  $|\mathcal{E}_1^k(X)| \leq 2(k-2)|X|$ .

*Proof.* Recall that every Type I range  $S'$  has a triple of points in  $X$  on its boundary. We call such a boundary point *middle point* of  $S'$  if its  $x$ -coordinate is between the  $x$ -coordinates of the other two boundary points. Moreover, if  $\ell$  denotes the vertical line through the middle point  $p$  of  $S'$  then  $S' \cap \ell$  is a vertical segment with an endpoint  $p$ . We call  $p$  a *lower*, respectively *upper*, middle point of  $S'$  if  $S' \cap \ell$  has  $p$  as its lower, respectively upper endpoint.

We remark that the middle point is well-defined, since no two points in  $X$  have the same  $x$ -coordinate. We shall show that every point of  $X$  is the lower, respectively upper, middle point of at most  $k-2$  Type I ranges. So let  $p$  be any point in  $X$  and  $\mathcal{S}_p$  be the set of all Type I ranges with lower middle point  $p$ . We want to show that  $t = |\mathcal{S}_p| \leq k-2$ . Then, by symmetry,  $p$  is also the upper middle point of at most  $k-2$  Type I ranges, which gives that the total number of Type I ranges is at most  $2(k-2)|X|$ .

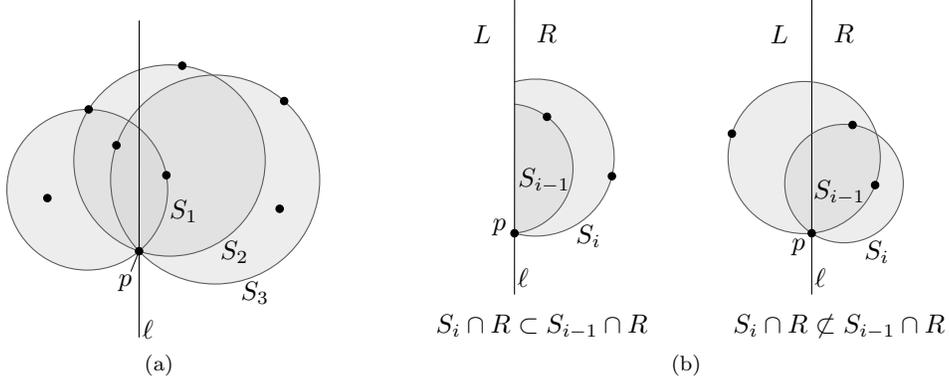


FIGURE 5. (a) A point  $p$  and the set  $\mathcal{S}_p$  of all Type I ranges with lower middle point  $p$ . (b) The two cases needed to prove (4).

Let  $\ell$  denote the vertical line through  $p$  and  $L, R$  denote the left and right halfplanes defined by  $\ell$ , respectively. Recall that  $L$  and  $R$  are open. Consider two ranges  $S_1, S_2$  from  $\mathcal{S}_p$ . We have that  $|X \cap S_1| = |X \cap S_2| = k$ . If  $S_1 \subseteq S_2$ , then the three points of  $X \cap \partial S_2$  must be in  $S_1$ , implying that  $|\partial S_1 \cap \partial S_2| \geq 3$ , a contradiction to Lemma 6 (i). Thus no two ranges from  $\mathcal{S}_p$  are included in each other. Since all these ranges contain  $p$  on the boundary, the boundaries of all these ranges “cross” at  $p$ . In particular, we can label the elements of  $\mathcal{S}_p$  by  $S_1, \dots, S_t$  such that going clockwise around  $p$  we see  $\ell, \partial S_1, \dots, \partial S_t, \ell, \partial S_1, \dots, \partial S_t$ . See Figure 5(a).

Let us focus on the right halfplane  $R$ . For convenience we define  $X_i = S_i \cap X \cap R$  for  $i = 1, \dots, t$  and  $X_0 = \emptyset$ . We claim that

$$|X_i \setminus X_{i-1}| - |X_{i-1} \setminus X_i| \geq 1 \quad \text{for } i = 1, \dots, t. \quad (4)$$

Indeed, if  $S_{i-1} \cap R \subset S_i \cap R$ , then  $X_{i-1} \setminus X_i = \emptyset$  whereas  $X_i \setminus X_{i-1}$  contains the point from  $X$  on  $\partial S_i \cap R$ . Otherwise,  $R$  contains a point from  $\partial S_i \cap \partial S_{i-1}$ , which together with  $p$  forms the two points in  $\partial S_i \cap \partial S_{i-1}$ . Thus by Lemma 6 (i) we have  $S_i \cap L \subset S_{i-1} \cap L$  and hence

$$|X_i \setminus X_{i-1}| = |X \cap (S_i \setminus S_{i-1})| = |X \cap (S_{i-1} \setminus S_i)| \geq |X_{i-1} \setminus X_i| + 1,$$

where the second equality holds because  $|X \cap S_i| = |X \cap S_{i-1}|$  and the last inequality holds because  $\partial S_{i-1}$  contains a point in  $(S_{i-1} \setminus S_i) \cap X$  in  $L$ . See Figure 5(b).

Since  $X_t = S_t \cap X \cap R$ ,  $|S_t \cap X| = k$  and  $(S_t \cap X) \setminus R$  contains at least two points, we have that  $|X_t| \leq k - 2$ . With the fact that  $|A \setminus B| - |B \setminus A| + |B \setminus C| - |C \setminus B| = |A \setminus C| - |C \setminus A|$  for any three sets  $A, B, C$  we can now conclude

$$k - 2 \geq |X_t| = |X_t \setminus X_0| - |X_0 \setminus X_t| = \sum_{i=1}^t (|X_i \setminus X_{i-1}| - |X_{i-1} \setminus X_i|) \stackrel{(4)}{\geq} t = |S_p|,$$

as desired.  $\square$

With Proposition 13 we can already prove a simple upper bound on the number of hyperedges in  $\mathcal{H}(X, S, k)$ .

**Proposition 14.**  $|\mathcal{E}(\mathcal{H}(X, S, k))| \leq 4k|X|$ .

*Proof.* For a hyperedge  $Y$ , consider an  $S$ -range  $S_Y$  with

$$Y \subseteq S_Y \quad \text{and} \quad |\partial S_Y \cap X| = 3 \quad \text{and} \quad |S_Y \cap X| \in \{k, k + 1\}. \quad (5)$$

If  $Y$  is a Type I hyperedge, then every representative range for  $Y$  has the properties (5). If  $Y$  is a Type II hyperedge and  $S_1$  is a representative range for  $Y$  with  $\partial S_1 \cap Y = \{p, q\}$ , then let  $L$  and  $R$  be the two halfplanes defined by  $\overline{pq}$ . Since  $(S_1 \Delta L) \cup (S_1 \Delta R) \supset \mathbb{R}^2 \setminus \overline{pq}$  at least one of  $\text{next}_L(S_1), \text{next}_R(S_1)$  exists by Lemma 8 (i), say  $\text{next}_L(S_1)$ . Moreover, since  $R \cap \partial S_1 \cap X = \emptyset$ , by (2)  $n_L(S_1)$  captures  $k$  or  $k + 1$  points from  $X$ . However, the former case is ruled out as  $Y$  has Type II, and hence we conclude that  $n_L(S_1)$  has the properties (5).

Now the set of all  $S_Y$  for all hyperedges  $Y$  is a subset of  $\mathcal{R}_1^k(X) \cup \mathcal{R}_1^{k+1}(X)$ . Thus by Proposition 13 we have at most  $2(k - 2)|X| + 2(k - 1)|X| \leq 4k|X|$  hyperedges.  $\square$

For a fixed  $k$ -suspension  $X'$  of  $X$  we define  $\hat{\mathcal{R}}_1^k(X')$  to be the set of all Type I ranges of  $\mathcal{H}(X', S, k)$  whose middle point lies in  $X$ .

**Proposition 15.** *Let  $X'$  be a fixed  $k$ -suspension of  $X$ . Then every point in  $X$  is the middle point of at least  $2(k - 2)$  Type I ranges in  $\mathcal{H}(X', S, k)$ . In particular,*

$$|\hat{\mathcal{R}}_1^k(X')| \geq 2(k - 2)|X|.$$

*Proof.* Let  $X$  be a point set with  $k$ -suspension  $X' = X \cup A \cup B \cup C$  and  $p \in X$  be any fixed point in  $X$ . Let  $\ell$  be the vertical line through  $p$  and  $L$  and  $R$  be the left and right open halfplanes defined by  $\ell$ , respectively.

We say that an  $S$ -range  $S'$  has *property*  $(a, b)$  if

- $p$  is the lower endpoint of the vertical segment  $\ell \cap S'$ ,
- $L \cap S'$  contains exactly  $a$  points from  $X'$ , one on its boundary if  $a \geq 1$ , and
- $R \cap S'$  contains exactly  $b$  points from  $X'$ , one on its boundary if  $b \geq 1$ .

We shall show that for each  $a = 1, \dots, k - 2$  there exists an  $S$ -range with property  $(a, k - 1 - a)$  of Type I with middle point  $p$ . This would imply that there are at least  $k - 2$  Type I ranges with lower middle point  $p$ . By symmetry, this would also imply that there are  $k - 2$  Type I ranges with upper middle point  $p$ , which makes in total  $2(k - 2)$  Type I ranges with middle point  $p$ , as desired. For that we shall first construct ranges of types  $(0, 0), (1, 0), \dots, (a, 0)$  and then ranges of types  $(a, 1), (a, 2), \dots, (a, k - 1 - a)$ .

So let  $p \in X$  and  $a \in \{1, \dots, k - 2\}$  be fixed. We start with any  $S$ -range  $S_0$  with property  $(0, 0)$ . Let  $q$  denote the upper endpoint of  $\ell \cap S_0$ . Then we define for  $i = 1, \dots, a$   $S_i$  to be

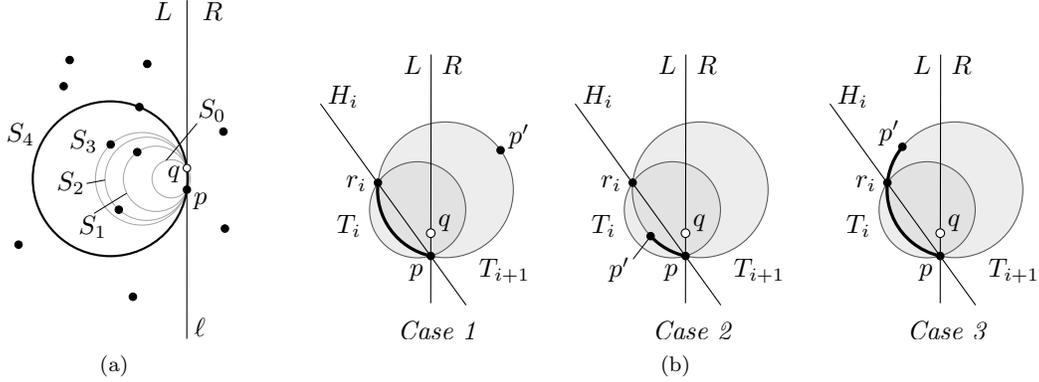


FIGURE 6. (a) The  $S$ -ranges  $S_0, \dots, S_4$  having property  $(0, 0), \dots, (4, 0)$ , respectively. (b) The three cases in the construction of  $T_{i+1}$  based on  $T_i$  and  $r_i$ . The component  $\gamma_{i+1}$  of  $\partial T_{i+1} \setminus \{p, r_{i+1}\}$  that is completely contained in  $L$  is drawn bold.

the next  $S$ -range of  $S_{i-1}$  in  $L$ . Clearly,  $S_i$  has property  $(i, 0)$ . In particular  $S_a$  has property  $(a, 0)$ . See Figure 6(a) for an illustration.

Next, we shall construct a sequence  $T_0, T_1, \dots$  of  $S$ -ranges and a sequence  $r_0, r_1, \dots$  of elements of  $X'$  such that **A**)  $r_i \in \partial T_i \cap L$ , **B**) the segment  $\ell \cap T_{i+1}$  is strictly longer than the segment  $\ell \cap T_i$ , **C**)  $|T_i \cap R \cap X'| = x_i$  with  $x_i \leq x_{i+1}$ , and **D**)  $|(T_i \setminus \gamma_i) \cap L \cap X'| = a$ , where  $\gamma_i$  denotes the component of  $\partial T_i \setminus \{p, r_i\}$  that is completely contained in  $L$ . Moreover, **E**) when  $x_i < x_{i+1}$ , then  $T_{i+1}$  has property  $(a, x_{i+1})$ , i.e.,  $T_{i+1}$  is a Type I range with middle point  $p$ . As a consequence, a subsequence of  $T_0, T_1, \dots$  consists of Type I ranges with middle point  $p$  and properties  $(a, 0), (a, 1), \dots, (a, k - a)$ .

Let  $T_0 = S_a$ ,  $r_0 \in \partial T_0 \cap (L \cap X')$ . Assume that  $T_0, \dots, T_i$  and  $r_0, \dots, r_i$  have been constructed. We define an  $S$ -range  $T_{i+1}$  based on  $T_i$  and  $r_i$  as follows. Let  $H_i$  denote the halfplane defined by  $\overline{pr_i}$  and containing  $q$ . We shall define  $T_{i+1}$  as the next  $S$ -range of  $T_i$  in  $H_i$ , i.e.,  $T_{i+1} = \text{next}_{H_i}(T_i)$ . The existence of this next range is guaranteed by Lemma 8 (i) and the property of suspension from Lemma 9. Note that Lemma 8 (ii) implies that the segment  $\ell \cap T_{i+1}$  is strictly longer than the segment  $\ell \cap T_i$ .

It remains to define the point  $r_{i+1}$ . Let  $p'$  be the unique point in  $(\partial T_{i+1} \setminus \partial T_i) \cap X'$ . We distinguish three cases.

*Case 1:*  $p' \in R$ . We have that  $|T_{i+1} \cap L \cap X'| = a$ ,  $|T_{i+1} \cap R \cap X'| = x_i + 1$  and  $\partial T_{i+1} \cap X' = \{r_i, p, p'\}$ . So  $T_{i+1}$  has property  $(a, x_{i+1})$  with  $x_{i+1} = x_i + 1$ . Set  $r_{i+1} = r_i$ , which implies  $\gamma_{i+1} \cap X' = \emptyset$ .

*Case 2:*  $p' \in L \setminus H_i$ . Then  $|T_{i+1} \cap L \cap X'| = a$  and  $|T_{i+1} \cap R \cap X'| = x_i$ , just like  $T_i$ . In this case we set  $r_{i+1} = p'$ , which gives again  $\gamma_{i+1} \cap X' = \emptyset$ .

*Case 3:*  $p' \in L \cap H_i$ . Then  $|T_{i+1} \cap L \cap X'| = a + 1$  and  $|T_{i+1} \cap R \cap X'| = x_i$ . We set  $r_{i+1} = p'$ , which implies  $r_i \in \gamma_{i+1}$  and hence  $|(T_{i+1} \setminus \gamma_{i+1}) \cap L \cap X'| = a$ .

We refer to Figure 6(b) for an illustration. We see that we either have  $r_{i+1} \neq r_i$  or  $x_{i+1} > x_i$ . Since there are finitely many possibilities for  $r_i$  and  $x_i$  and no pair  $\{r_i, x_i\}$  occurs twice, at some point we obtain a range of type  $(a, k - 1 - a)$  that is a Type I range with middle point  $p$ .  $\square$

#### 4.2. Relation between the number of Type I and Type II ranges.

Recall that for a fixed Type I hyperedge  $Y$  in  $\mathcal{H}(X, S, k)$  we denote by  $\mathcal{R}(Y)$  the set of representative ranges for  $Y$ . Recall also that  $\hat{\mathcal{R}}_1^k(X')$  is the set of all Type I ranges whose middle point lies in  $X$ . We denote by  $\hat{\mathcal{E}}_1^k = \hat{\mathcal{E}}_1^k(X')$  the subset of hyperedges in  $\mathcal{H}(X', S, k)$  that have at least one representative  $S$ -range in  $\hat{\mathcal{R}}_1^k(X')$ .

**Lemma 16.** *For any fixed  $k$ -suspension  $X'$  of  $X$  we have*

$$3|\hat{\mathcal{R}}_1^k(X')| + 2|\mathcal{R}_2^k(X)| \leq 3|\mathcal{R}_1^{k+1}(X')| + 2 \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1).$$

*Proof.* Consider the set  $P$  of all ordered pairs  $(S_1, S_2)$  of  $S$ -ranges, such that

- (A)  $S_1 \in \hat{\mathcal{R}}_1^k(X') \cup \mathcal{R}_2^k(X)$ .
- (B)  $\partial S_1 \cap \partial S_2$  is a pair  $p, q$  of points in  $X'$ .
- (C)  $S_2$  is the next range of  $S_1$  in a halfplane defined by  $\overline{pq}$ .
- (D)  $X' \cap S_1 \subseteq X' \cap S_2$ .

We shall bound  $|P|$  in two different ways. We define  $P_1$  and  $P_2$  to be the subsets of  $P$  containing all pairs  $(S_1, S_2)$  with  $S_1 \in \hat{\mathcal{R}}_1^k(X')$  and  $S_1 \in \mathcal{R}_2^k(X)$ , respectively. Further, let  $P_3$  and  $P_4$  be the subsets of  $P$  containing all pairs  $(S_1, S_2)$  with  $|S_2 \cap X'| = k + 1$  and  $|S_2 \cap X'| = k$ , respectively.

For a pair  $(S_1, S_2) \in P$  we say that  $S_2$  is an image of  $S_1$  and  $S_1$  is a preimage of  $S_2$ . Note that, if  $S_2$  is an image of  $S_1$ , then  $S_1$  contains  $k$  points from  $X'$  and thus  $S_2$  contains either  $k$  or  $k + 1$  points from  $X'$ . In the former case,  $S_1$  and  $S_2$  are distinct representative  $S$ -ranges for the same Type I hyperedge in  $\mathcal{H}(X', S, k)$ , see Figure 7(a), while in the latter case  $S_2$  is a Type I range in  $\mathcal{H}(X', S, k + 1)$ , see Figure 7(a) and (b). Then we see that

$$P_4 = \bigcup_{Y \in \hat{\mathcal{E}}_1^k} \{(S_1, S_2) \in P \mid S_1, S_2 \in \mathcal{R}(Y)\}$$

and  $P = P_1 \dot{\cup} P_2$  and  $P = P_3 \dot{\cup} P_4$ .

We shall show that, on one hand,  $|P_1| = 3|\hat{\mathcal{R}}_1^k(X')|$  and  $|P_2| = 2|\mathcal{R}_2^k(X)|$ , while on the other hand,  $|P_3| \leq 3|\mathcal{R}_1^{k+1}(X')|$  and  $|P_4| \leq 2 \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1)$ , which in turn will imply the desired inequality.

To prove that  $|P_1| = 3|\hat{\mathcal{R}}_1^k(X')|$ , consider  $S_1 \in \hat{\mathcal{R}}_1^k(X')$  with middle point  $p_0 \in X$ . Let  $p_1, p_2$  be the two points in  $X' \cap \partial S_1$  different from  $p_0$ . For  $i = 0, 1, 2$  let  $H_i$  be the halfplane defined by  $\overline{p_{i-1}p_{i+1}}$  containing  $p_i$ , where indices are taken modulo 3. By (C) and (D)  $S_2$  is the next  $S$ -range of  $S_1$  in  $H_i$  for some  $i \in \{0, 1, 2\}$ . By Lemma 8 (i) the next  $S$ -range is unique and it exists whenever  $H_i \setminus S_1$  contains a point from  $X'$ . This point exists from the definition of suspension. Hence  $S_1$  has exactly three images.

To prove that  $|P_2| = 2|\mathcal{R}_2^k(X)|$ , consider  $S_1 \in \mathcal{R}_2^k(X)$ . Let  $X \cap \partial S_1 = \{p_1, p_2\}$ . An image of  $S_1$  is the next  $S$ -range of  $S_1$  in a halfplane defined by  $\overline{p_1 p_2}$ . Hence  $S_1$  has at most two images corresponding to each half plane. And similarly to the previous case, both halfplanes contain at least one point of  $X' \setminus S_1$ . Hence  $S_1$  has exactly two images. Further, by definition  $\partial S_2 \cap X'$  contains a third point  $q$  different from  $p_1, p_2$ , i.e.,  $S_2$  is a Type I range. Since  $S_1$  is a Type II range we have  $q \notin S_1$  and  $S_2 \in \mathcal{R}_1^{k+1}(X')$ .

To prove that  $|P_3| \leq 3|\mathcal{R}_1^{k+1}(X')|$ , we shall show that every  $S_2 \in \mathcal{R}_1^{k+1}(X')$  has at most three preimages. Let  $X' \cap \partial S_2 = \{p_0, p_1, p_2\}$ , and let  $H_i$  and  $\bar{H}_i$  be the halfplanes defined by  $\overline{p_{i-1}p_{i+1}}$  containing  $p_i$  and *not* containing  $p_i$ , respectively, where indices are taken modulo 3 again. By (B), (C) and (D) every preimage  $S_1$  of  $S_2$  corresponds to a point  $p_i \in \{p_0, p_1, p_2\}$

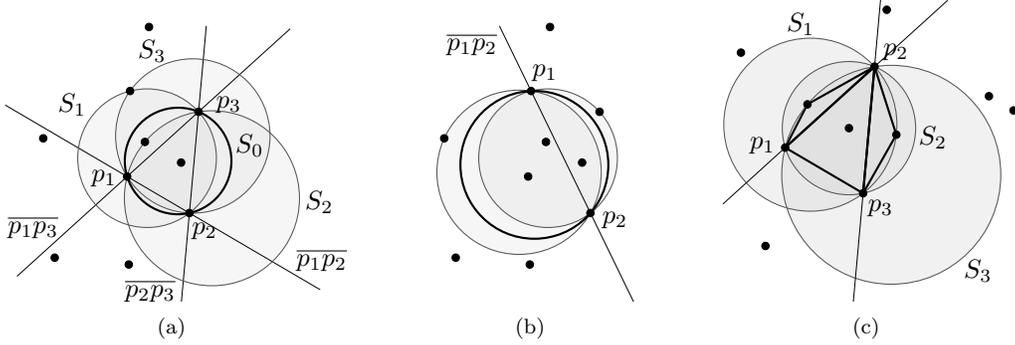


FIGURE 7. (a) The three images  $S_1, S_2, S_3$  of a Type I range  $S_0$  (in bold). Note that  $S_1$  and  $S_3$  correspond to the same hyperedge in  $\mathcal{H}(X', S, k+1)$ , and that  $S_0$  and  $S_2$  correspond to the same hyperedge in  $\mathcal{H}(X', S, k)$ . (b) The two images of a Type II range (in bold); one being also an image of the Type I range in (a). (c) Three representative ranges for the same hyperedge  $Y$  and the outerplanar graph  $G(Y)$  (in bold). Here  $(S_1, S_2) \in P_4$  and  $(S_2, S_1) \in P_4$ , both with respect to  $\{p_2, p_3\}$ , as well as  $(S_2, S_3) \in P_4$  and  $(S_3, S_2) \in P_4$ , both with respect to  $\{p_1, p_2\}$ .

with  $p_i \in S_2 \setminus S_1$ , such that  $S_2$  is the next  $S$ -range of  $S_1$  in the halfplane  $H_i$ . Hence  $S_2$  has at most three preimages.

Indeed, if  $S' = \text{next}_{\bar{H}_i}(S_2)$  exists and captures  $k$  points from  $X'$ , then  $(S', S_2) \in P_3$ , provided the middle point of  $S'$  lies in  $X$ . Whereas, if  $S'$  captures  $k+1$  points from  $X'$ , then  $Y = X' \cap S' \cap S_2$  is a Type II hyperedge and for its representative range  $S''$  we have  $(S'', S_2) \in P_3$ , provided  $Y \subset X$ . Finally,  $S'$  can not capture  $k+2$  points since  $p_i \in S_2 \setminus S'$ .

To prove that  $|P_4| \leq 2 \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1)$ , we shall show that for every hyperedge  $Y \in \hat{\mathcal{E}}_1^k$  there exist at most  $2(|\mathcal{R}(Y)| - 1)$  ordered pairs  $(S_1, S_2) \in P$  with  $Y = X' \cap S_1 = X' \cap S_2$ . Consider the graph  $G(Y)$  defined above, whose vertex set is  $Y$  and whose edges are all pairs  $\{p, q\} \subseteq Y$  such that  $p, q \in \partial S'$  for a representative range  $S' \in \mathcal{R}(Y) \subset \mathcal{R}_1^k(X')$ . By Lemma 12 (i) connecting any two points in  $Y$  that are adjacent in  $G(Y)$  with a straight line segment gives an outerplanar drawing of  $G(Y)$ .

Now if  $(S_1, S_2) \in P_4$ , then  $\partial S_1 \cap \partial S_2 = \{p, q\}$  is an inner edge of  $G(Y)$ , see Figure 7(c). Moreover we claim that at most two ranges in  $\mathcal{R}(Y)$  have  $p$  and  $q$  on their boundary. Indeed, if there were three such ranges  $S_1, S_2, S_3$ , then after relabeling we have  $S_3 \cap R \subset S_1 \cap R$  and  $S_3 \cap L \subset S_2 \cap L$ , by Lemma 6 (ii), where  $R$  and  $L$  are the two halfplanes defined by  $p, q$ . With  $Y = S_i \cap X'$  for  $i = 1, 2, 3$  this would imply that  $\partial S_3 \cap X = \{p, q\}$  – a contraction to  $S_3 \in \mathcal{R}(Y)$  and the fact that  $Y$  has Type I.

Thus, every pair  $(S_1, S_2) \in P_4$  gives rise to an inner edge of  $G(Y)$  and every inner edge  $\{p, q\}$  of  $G(Y)$  gives at most two ordered pairs in  $P_4$ . Because by Lemma 12 (i)  $G(Y)$  has at most  $|\mathcal{R}(Y)| - 1$  inner edges, we have the desired inequality.

Now we conclude that  $|P| = |P_1| + |P_2| = 3|\hat{\mathcal{R}}_1^k(X')| + 2|\mathcal{R}_2^k(X)|$ , whereas  $|P| = |P_3| + |P_4| \leq 3|R_1^{k+1}(X')| + 2 \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1)$ . Together this gives the claimed inequality.  $\square$

## 5. PROOF OF THEOREM 3.

*Proof of Theorem 3.* For  $k = 1$  and any  $X$ , the hypergraph  $\mathcal{H}(X, S, k)$  clearly has at most  $|X| = (2k - 1)|X|$  hyperedges. So we may assume that  $k \geq 2$ .

Consider an  $S$ -capturing hypergraph on a point set  $X$  for  $S$  closed and compact. By Lemma 7, we can assume that  $S$  is a nice shape and  $X$  is in a general position with respect to  $S$ . Let us consider a fixed  $k$ -suspension  $X' = X \cup A \cup B \cup C$  of  $X$ . Then

$$\mathcal{R}_1^k(X) \subseteq \hat{\mathcal{R}}_1^k(X') \quad \text{and} \quad \mathcal{E}_1^k(X) \subseteq \hat{\mathcal{E}}_1^k. \quad (6)$$

Moreover, by Lemma 16 we have

$$3|\hat{\mathcal{R}}_1^k(X')| + 2|\mathcal{R}_2^k(X)| \leq 3|\mathcal{R}_1^{k+1}(X')| + 2 \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1). \quad (7)$$

By Proposition 13 we have

$$|\mathcal{R}_1^{k+1}(X')| \leq 2(k-1)(n+3k) \leq 2(k-1)n + O(k^2), \quad (8)$$

and by Lemma 15 we have

$$|\hat{\mathcal{R}}_1^k(X')| \geq 2(k-2)n. \quad (9)$$

Putting (3), (6), (7), (8) and (9) together, we conclude that

$$\begin{aligned} |\mathcal{E}_1^k(X)| + |\mathcal{E}_2^k(X)| &\stackrel{(3),(6)}{\leq} |\hat{\mathcal{R}}_1^k(X')| - \sum_{Y \in \hat{\mathcal{E}}_1^k} (|\mathcal{R}(Y)| - 1) + |\mathcal{R}_2^k(X)| \\ &\stackrel{(7)}{\leq} \frac{3|\mathcal{R}_1^{k+1}(X')| - |\hat{\mathcal{R}}_1^k(X')|}{2} \\ &\stackrel{(8),(9)}{\leq} \frac{6(k-1)n - 2(k-2)n}{2} + O(k^2) = (2k-1)n + O(k^2), \end{aligned}$$

as desired.  $\square$

## 6. CONCLUSIONS AND REMARKS

In this paper we investigated  $k$ -uniform hypergraphs whose vertex set  $X$  is a set of points in the plane and whose hyperedges are exactly those  $k$ -subsets of  $X$  that can be captured by a homothetic copy of a fixed convex compact set  $S$ . These are so called  $k$ -uniform  $S$ -capturing hypergraphs. We have shown that every such hypergraph has at most  $(2k-1)|X| + O(k^2)$  hyperedges and that for every (nice) shape  $S$  there are hypergraphs with  $(2k-1)|X| - O(k^2)$  hyperedges.

To prove the upper bound we partitioned the hyperedges into two kinds: Type I hyperedges can be captured by a range containing three points on its boundary, while Type II hyperedges can not. We showed that there are at most  $2(k-1)|X|$  hyperedges of Type I. Then we showed, using an indirect proof, that the total number of hyperedges of both types is at most  $(2k-1)|X| + O(k^2)$ . We also proved that there are always (if  $X$  is suspended) at least  $2(k-1)|X|$  Type I hyperedges, which together implies that there are at most  $3|X| + O(k^2)$  hyperedges of Type II.

However, we have no direct proof of the fact that there are at most  $3|X| + O(k^2)$  hyperedges of Type II. We may even ask a stronger question.

**Problem 17.** Is it true that for every  $X$ ,  $S$  and  $k$  the Type II hyperedges of  $\mathcal{H}(X, S, k)$  correspond bijectively to the edges of some planar graph on  $X$ ?

Note that when  $k = 2$ , then this is indeed the case, since  $\mathcal{H}(X, S, 2)$  is a planar graph itself.

As an immediate corollary of our  $(2k - 1)|X| + O(k^2)$  upper bound we obtain a bound on the number of hyperedges of size *at most*  $k$ : For every point set  $X$ , every convex set  $S$  and every  $k \geq 2$  at most  $k^2|X| + O(k^3)$  different subsets of  $X$  of size at most  $k$  can be captured by a homothetic copy of  $S$ . This refines the recent  $O(k^2|X|)$  bound by Buzaglo, Pinchasi and Rote [2].

Tight bounds are only known in case  $k = 2$ . Indeed, every 2-uniform  $S$ -capturing hypergraph is a planar graph [5], called the *convex distance function Delaunay graph*, and thus has at most  $3n - 6$  edges. We suspect that our methods can be improved to give a tight bound for arbitrary  $k \geq 2$ .

**Problem 18.** What is the maximum number of hyperedges in a  $k$ -uniform  $S$ -capturing hypergraph when  $S$  is a fixed convex compact set? In particular, is the correct answer given by  $(2k - 1)|X| - 2(k - 1)(2k - 1) = (2k - 1)|X| - 2\binom{2k-1}{2}$ ?

Another interesting open problem concerns topological hypergraph defined by a family of pseudodiscs. Here, the vertex set  $X$  is again a finite point set in the plane and every hyperedge is a subset of  $X$  surrounded by a closed Jordan curve such that any two such curves have at most two points in common. Buzaglo, Pinchasi and Rote [2] prove that every pseudodisc topological hypergraph at most  $O(k^2|X|)$  hyperedges of size *at most*  $k$ .

**Problem 19.** What is the maximum number of hyperedges of size *exactly*  $k$  in a pseudodisc topological hypergraph?

#### ACKNOWLEDGMENTS

We would like to thank Günter Rote and János Pach for helpful remarks and referring us to relevant literature.

#### REFERENCES

- [1] Peter Brass, William Moser, and János Pach. *Research Problems in Discrete Geometry*. Springer, 2005.
- [2] Sarit Buzaglo, Rom Pinchasi, and Günter Rote. Topological hypergraphs. In *Thirty Essays on Geometric Graph Theory*, pages 71–81. Springer, 2013.
- [3] Jean Cardinal, Kolja B. Knauer, Piotr Micek, and Torsten Ueckerdt. Making triangles colorful. *CoRR*, abs/1212.2346, 2012.
- [4] Xiaomin Chen, János Pach, Mario Szegedy, and Gábor Tardos. Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles. *Random Struct. Algorithms*, 34(1):11–23, 2009.
- [5] L. Paul Chew and Robert L. (Scot) Dyrsdale III. Voronoi diagrams based on convex distance functions. In *Proceedings of the first annual symposium on Computational geometry (SoCG 85)*, pages 235–244. ACM, 1985.
- [6] Peter Frankl and János Pach. On disjointly representable sets. *Combinatorica*, 4(1):39–45, 1984.
- [7] Balázs Keszegh and Dömötör Pálvölgyi. Octants are cover-decomposable into many coverings. *CoRR*, abs/1207.0672, 2012.
- [8] Seog-Jin Kim, Alexandr Kostochka, and Kittikorn Nakprasit. On the chromatic number of intersection graphs of convex sets in the plane. *Electron. J. Combin.*, 11(1):R52, 2004. 12 pp.
- [9] Seog-Jin Kim, Kittikorn Nakprasit, Michael J. Pelsmajer, and Jozef Skokan. Transversal numbers of translates of a convex body. *Discrete Mathematics*, 306(18):2166–2173, 2006.
- [10] Jirí Matoušek. *Lectures on Discrete Geometry*. Springer, 2002.
- [11] János Pach. Covering the plane with convex polygons. *Discrete & Computational Geometry*, 1:73–81, 1986.
- [12] János Pach and Micha Sharir. *Combinatorial geometry and its algorithmic applications: The Alcalá lectures*. Number 152. American Mathematical Soc., 2009.

- [13] János Pach and Gábor Tardos. Coloring axis-parallel rectangles. *J. Comb. Theory, Ser. A*, 117(6):776–782, 2010.
- [14] János Pach and Gábor Tardos. Tight lower bounds for the size of epsilon-nets. In *Proc. Symposium on Computational Geometry (SoCG)*, pages 458–463, 2011.
- [15] János Pach and Gerhard Woeginger. Some new bounds for  $\varepsilon$ -nets. In *Proceedings of the sixth annual symposium on Computational geometry*, pages 10–15. ACM, 1990.
- [16] Dömötör Pálvölgyi. Indecomposable coverings with unit discs. *arXiv preprint arXiv:1310.6900*, 2013.
- [17] Dömötör Pálvölgyi and Géza Tóth. Convex polygons are cover-decomposable. *Discrete & Computational Geometry*, 43(3):483–496, 2010.
- [18] Norbert Sauer. On the density of families of sets. *J. Combinatorial Theory Ser. A*, 13:145–147, 1972.
- [19] Walter Schnyder. Embedding planar graphs on the grid. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms (SODA 90)*, pages 138–148. Society for Industrial and Applied Mathematics, 1990.
- [20] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific J. Math.*, 41:247–261, 1972.
- [21] Shakhar Smorodinsky. Conflict-free coloring and its applications. *arXiv preprint arXiv:1005.3616v3*, 2012.
- [22] Gábor Tardos and Géza Tóth. Multiple coverings of the plane with triangles. *Discrete & Computational Geometry*, 38(2):443–450, 2007.
- [23] Vladimir Vapnik and Alexey Červonenkis. The uniform convergence of frequencies of the appearance of events to their probabilities. *Teor. Veroyatnost. i Primenen.*, 16:264–279, 1971.
- [24] David R Wood. Hypergraph colouring and degeneracy. *arXiv preprint arXiv:1310.2972*, 2013.

## APPENDIX

For a set  $S$ , we call a line  $\ell$  touching  $S$  if  $\ell$  intersects  $S$  in exactly one point. So, for each point on the boundary of a nice shape there is exactly one touching line intersecting  $S$  at this point.

**Lemma 6.** *If  $S$  is a nice shape,  $S_1$  and  $S_2$  are distinct  $S$ -ranges, then each of the following holds.*

- (i)  $\partial S_1 \cap \partial S_2$  is a set of at most two points.
- (ii) If  $\partial S_1 \cap \partial S_2 = \{p, q\}$  and  $L$  and  $R$  are the two open halfplanes defined by  $\overline{pq}$ , then
  - $S_1 \cap L \subset S_2 \cap L$  and  $S_1 \cap R \supset S_2 \cap R$  or
  - $S_1 \cap L \supset S_2 \cap L$  and  $S_1 \cap R \subset S_2 \cap R$ .
- (iii) Any three non-collinear points lie on the boundary of a unique  $S$ -range.
- (iv) For a subset of points  $X \subset \mathbb{R}^2$  and any  $Y \subset X$ ,  $|Y| \geq 2$ , that is captured by some  $S$ -range there exists at least one  $S$ -range  $S'$  with  $Y = X \cap S'$  and  $|\partial S' \cap X| \geq 2$ .

*Proof.* (i) We show that for any two  $S$ -ranges  $S_1, S_2$  such that  $\{p, q, r\} \subseteq \partial S_1 \cap \partial S_2$ , for distinct  $p, q, r$ ,  $S_1$  coincides with  $S_2$ . Assume not, consider a homothetic maps  $f_1, f_2$  from  $S_1, S_2$  to  $S$ , respectively. Let  $p_i, q_i, r_i$  be the images of  $p, q, r$  under  $f_i$ ,  $i = 1, 2$ . Then we have that  $p_1, q_1, r_1$  and  $p_2, q_2, r_2$  form congruent triangles  $T_1, T_2$  with vertices on  $\partial S$ . If these two triangles coincide, then  $S_1 = S_2$ . Otherwise consider two cases: a corner of one triangle is contained in the interior of the other triangle or not. If, without loss of generality, a corner of  $T_1$  is in the interior of  $T_2$ , then by convexity, this corner can not be on  $\partial S$ . Otherwise,  $T_1$  and  $T_2$  are either disjoint or share a point on the corresponding side. In either case, one of the sides of  $T_1$  is on the same line as the corresponding side of  $T_2$ , otherwise convexity of  $S$  is violated. Then, the boundary of  $S$  contains 3 collinear points, a contradiction to the fact that  $S$  is a nice shape.

- (ii) We have, without loss of generality that  $S_1 \cap L \subset S_2 \cap L$ . If  $S_1 \cap R \supset S_2 \cap R$ , we are done. Otherwise, we have that  $S_1 \cap R \subset S_2 \cap R$ , and, in particular  $S_1 \subseteq S_2$ . Note that

at  $p$  and  $q$  the ranges  $S_1$  and  $S_2$  have the same touching lines. Indeed, these lines are unique since  $S$  is nice. Consider maps  $f_1$  and  $f_2$  as before. We see that  $p, q$  are mapped into the same pair of points under both maps. Thus  $S_1 = S_2$ , a contradiction.

The remaining two items can be proven by considering two points  $p, q$  in the plane and the set  $\mathcal{S}(p, q)$  of all  $S$ -ranges  $S'$  with  $p, q \in \partial S'$ . Indeed, given fixed  $p$  and  $q$  there is a bijection  $\phi$  between the  $S$ -ranges in  $\mathcal{S}(p, q)$  and the set  $\mathcal{L}(p, q)$  of lines whose intersection with  $S$  is a non-trivial line segment parallel to the line  $\overline{pq}$ . We refer to Figure 8(a) for an illustration.

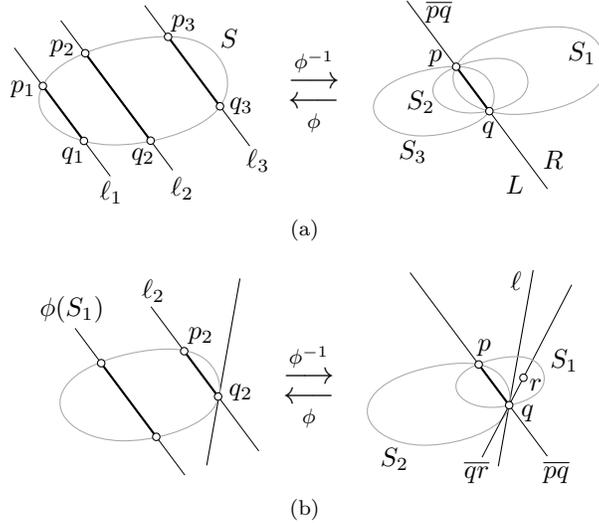


FIGURE 8. (a) A nice shape  $S$ , two points  $p, q$  in the plane, three lines  $\ell_1, \ell_2, \ell_3 \in \mathcal{L}(p, q)$  and the corresponding  $S$ -ranges  $S_1, S_2, S_3 \in \mathcal{S}(p, q)$ . (b) Given an  $S$ -range  $S_1 \in \mathcal{S}(p, q)$  with  $r \in S_1$  we can find an  $S$ -range  $S_2 \in \mathcal{S}(p, q)$  with  $r \notin S_2$ .

We verify that this bijection exists: For a line  $\ell' \in \mathcal{L}(p, q)$  such that  $\ell' \cap \partial S = \{p_1, q_1\}$ , let  $S'$  be a range obtained by contracting and translating  $S$  such that  $p_1$  and  $q_1$  are mapped into  $p$  and  $q$ , respectively. Let  $\phi^{-1}(\ell') = S'$ . Given an  $S'$ -range with  $p, q \in \partial S'$ , consider a translation and contraction that maps  $S'$  to  $S$ . Let  $p_1$  and  $q_1$  be the images of  $p$  and  $q$  under this transform, then let  $\phi(S') = \ell$ , where  $\ell$  is a line through  $p_1$  and  $q_1$ . See again Figure 8(a) for an illustration.

(iii) Consider any three non-collinear points  $p, q, r$  in the plane. We shall first show that there is an  $S$ -range with  $p, q, r$  on its boundary. Let  $S_1$  be an  $S$ -range of smallest area containing all three points. Clearly,  $|\partial S_1 \cap \{p, q, r\}|$  is either 2 or 3. In the latter case we are done. So assume without loss of generality that  $\partial S_1 \cap \{p, q, r\} = \{p, q\}$ .

Now, we claim that there is another  $S$ -range  $S_2$  that contains  $p$  and  $q$  but not  $r$ , with  $p, q \in \partial S_2$ . To find  $S_2$ , containing  $p, q$  on its boundary and not containing  $r$ , consider the triangle  $p, q, r$  and a line  $\ell$  that goes through  $q$  having  $p$  and  $r$  on the different sides. Assume without loss of generality that  $q$  is the lowest point among  $p, q, r$ , and that  $\overline{qr}$  and  $\ell$  have positive slopes. Next, let  $q_2$  be a point on  $\partial S$  whose touching line is parallel to  $\ell$  and such that  $S$  is above this touching line. See Figure 8(b). Let  $\ell_2$  be the line

parallel to  $\overline{pq}$  and containing  $q_2$  and let  $p_2$  be the unique point in  $\partial S \cap \ell_2$  different from  $q_2$ . Note that  $|\ell_2 \cap \partial S| = 2$  follows from the fact that  $S$  is a nice shape and  $\ell_2$  and  $\ell$  have different slopes. Let  $S_2 = \phi^{-1}(\ell_2)$ . We see that  $S_2$  is above  $\ell$ , but  $r$  is below  $\ell$ . So,  $r \notin S_2$ .

Since  $\phi$  is continuous and any two lines in  $\mathcal{L}(p, q)$  can be continuously transformed into each other within  $\mathcal{L}(p, q)$ , we conclude that  $S_1$  can be continuously transformed into  $S_2$  within  $\mathcal{S}(p, q)$ . Thus, there is an  $S$ -range  $S' \in \mathcal{S}(p, q)$  such that  $r \in \partial S'$ . This proves that any three non-collinear points  $p, q, r$  lie on the boundary of some  $S$ -range. The uniqueness of such a range follows from (i).

- (iv) Let  $Y$  be a hyperedge in  $\mathcal{H}(X, S, k)$  and  $S_1$  be an  $S$ -range capturing  $Y$ . Contract  $S_1$  until the resulting range  $S_2$  contains at least one point,  $p$  of  $Y$ , on its boundary. If  $|\partial S_2 \cap Y| \geq 2$ , we are done. Otherwise, consider a small  $S$ -range  $S_3$  containing  $p$  on its boundary and not containing any other points of  $X$ . Let  $q$  be the second point in  $\partial S_2 \cap \partial S_3$ . Similarly to the previous argumentation,  $S_2$  can be continuously transformed into  $S_3$  within  $\mathcal{S}(p, q)$ . Each intermediate  $S$ -range is contained in  $S_2 \cup S_3$  and thus contains no points of  $X \setminus Y$ . One of the intermediate  $S$ -ranges will contain another point of  $Y$  on its boundary. □

**Lemma 7.** *For any point set  $X$ , positive integer  $k$  and a convex compact set  $S$ , there is a point set  $X'$  and a nice shape  $S'$ , such that  $|X'| = |X|$  and the number of edges in  $\mathcal{H}(X', S', k)$  is at least as large as the number of edges in  $\mathcal{H}(X, S, k)$ .*

*Proof.* First, we shall modify  $S$  slightly. Consider all the hyperedges of  $\mathcal{H}(X, S, k)$  and for each of them choose a single capturing range. Recall that two  $S$ -ranges are contraction/dilation of one another if in their corresponding homothetic maps the origin is mapped to the same point. For each hyperedge consider two distinct capturing  $S$ -ranges  $S_1, S_2$  with  $S_2$  being a dilation of  $S_1$ . Among all hyperedges, consider the one for which these two  $S$ -ranges,  $S_1 \subset S_2$  are such that the stretching factor between  $S_1$  and  $S_2$  is the smallest.

Let  $S'$  be a nice shape,  $S_1 \subset S' \subset S_2$ . Replace each of the other capturing ranges with an appropriate  $S'$ -range. Now, we have that  $\mathcal{H}(X, S', k)$  has at least as many hyperedges as  $\mathcal{H}(X, S, k)$ . Next, we shall move the points from  $X$  slightly so that the new set  $X'$  is in a general position with respect to  $X'$  and contains as many hyperedges as  $\mathcal{H}(X, S', k)$ .

Observe first that since  $X$  is a finite point set, we can move each point of  $X$  by some small distance, call it  $\epsilon(X)$  in any direction such that the resulting hypergraph has the same set of hyperedges as  $\mathcal{H}(X, S', k)$ .

Call a point  $x \in X$  *bad* if either  $x$  is on a vertical line together with some other point of  $X$ ,  $x$  is on a line with two other points of  $X$ , or  $x$  is on the boundary of an  $S'$ -range together with at least three other vertices of  $X$ .

We shall move a bad  $x$  such that a new point set has smaller number of bad points and such that the resulting hypergraph has at least as many edges as  $\mathcal{H}(X, S', k)$ . From a ball  $B(x, \epsilon(X))$  delete all vertical lines passing through a point of  $X$ , delete all lines that pass through at least two points of  $X$  and delete all boundaries of all  $S'$ -ranges containing at least 3 points of  $X$ . All together we have deleted at most  $n + \binom{n}{2} + \binom{n}{3}$  curves because there is one vertical line passing through each point, at most  $\binom{n}{2}$  lines passing through some two points of  $X$  and at most  $\binom{n}{3}$   $S'$ -ranges having some three points of  $X$  on their boundary. So, there are points left in  $B(x, \epsilon(X))$  after this deletion. Replace  $x$  with an available point  $x'$  in  $B(x, \epsilon(X))$ . Observe that  $x'$  is no longer bad in a new set  $X - \{x\} \cup \{x'\}$ . Moreover, if

$z \in X$ ,  $z \neq x$  was not a bad point, it is not a bad point in a new set  $X - \{x\} \cup \{x'\}$ . Indeed, since  $x'$  is not on a vertical line with any other point of  $X$  and not on any line containing two points of  $X$ ,  $z$  is not on a bad line with  $x'$ . Moreover, since  $x'$  is not on the boundary of an  $S'$ -range together with at least three other points of  $X$ ,  $z$  can not be together with  $x'$  on the boundary of a  $S'$ -range containing more than 3 points of  $X$  on its boundary.  $\square$

**Lemma 8.** *Let  $S_1$  be any  $S$ -range,  $p, q$  be two distinct points on  $\partial S_1$ , and  $R, L$  be the two halfplanes defined by  $\overline{pq}$ . Then the following holds.*

- (i) *If  $(S_1 \triangle L) \cap X \neq \emptyset$  then  $\text{next}_L(S_1)$  exists and is unique.*
- (ii) *If  $S_2 = \text{next}_L(S_1)$  then  $S_2 \cap R \subset S_1 \cap R$ .*

*Proof.* (i) Recall that given a convex set  $S$  and two points  $p, q$ , we denote by  $\mathcal{S}(p, q)$  the set of all  $S$ -ranges with  $p$  and  $q$  on their boundary and by  $\mathcal{L}(p, q)$  the set of all lines whose intersection with  $S$  is a non-trivial line segment parallel to  $\overline{pq}$ . For a finite point set  $X$  let us consider the finite subset  $\mathcal{S}^*(p, q)$  of  $\mathcal{S}(p, q)$  of  $S$ -ranges that contain a point from  $X \setminus \{p, q\}$  on their boundary. Via the bijection  $\phi$  between  $\mathcal{S}(p, q)$  and  $\mathcal{L}(p, q)$  we obtain a corresponding finite subset  $\mathcal{L}^*(p, q)$  of  $\mathcal{L}(p, q)$ .

Now given any  $S_1 \in \mathcal{S}(p, q)$  the next  $S$ -range  $S_2$  of  $S_1$  in the upper halfplane  $H$  defined by  $\overline{pq}$  corresponds to the line  $\phi(S_2) = \ell_2 \in \mathcal{L}^*(p, q)$  that lies in the lower halfplane defined by  $\ell_1 = \phi(S_1)$  and is closest to  $\ell_1$ . Clearly the line  $\ell_2$ , and hence also the  $S$ -range  $S_2$ , is uniquely defined whenever  $S_1 \triangle H$  contains a point from  $X$ .

- (ii) Without loss of generality let  $\overline{pq}$  be vertical and  $L$  be the left halfplane defined by  $\overline{pq}$ . Then  $\phi(S_1), \phi(S_2)$  are two vertical lines, with  $\phi(S_1)$  to the left of  $\phi(S_2)$ . This immediately implies that  $S_2 \cap R \subset S_1 \cap R$ .  $\square$

**Lemma 12.**

- (i) *For a hyperedge  $Y$  of Type I,  $G(Y)$  is outerplanar and the number of inner edges of  $G(Y)$  is at most  $|\mathcal{R}(Y)| - 1$ .*
- (ii) *All Type II representative  $S$ -ranges for the same hyperedge have the same pair of  $X$  in their boundary.*

*Proof.*

- (i) Note that each element in  $\mathcal{R}(Y)$  corresponds to a triangle in  $G = G(Y)$ . To show that  $G$  is outerplanar, let  $Y' \subseteq Y$  be the set of corners of the convex hull of  $Y$ . Since  $S$  is convex every point  $x \in Y \setminus Y'$  is an isolated vertex in  $G$ . So it is enough to find an outerplanar drawing of  $G[Y']$ . To this end, it suffices to draw every vertex at the position of its corresponding point and every edge as a straight-line segment.

Assuming that two edges  $x_1y_1, x_2y_2 \in E(G)$  cross, without loss of generality these four vertices appear around the convex hull of  $Y$  in the clockwise order  $x_1, x_2, y_1, y_2$ . But then the  $S$ -ranges  $S_1$  and  $S_2$  with  $Y \subset S_i$  and  $\{x_i, y_i\} = \partial S_i \cap Y$  for  $i = 1, 2$  have at least four intersections on their boundaries. See Figure 9(c) for an illustration. This is a contradiction to Lemma 6 (i), i.e., that  $|\partial S_1 \cap \partial S_2| \leq 2$ .

Finally recall that in edge in  $G(Y)$  is an inner edge if it is contained in at least two triangles. Since  $G(Y)$  is outerplanar, every inner edge corresponds to an edge bounding two inner triangular faces in an outerplanar drawing of  $G(Y)$ . Since the subgraph of the plane dual graph induced by all triangular faces is a forest, and there are at least  $|\mathcal{R}(Y)|$  triangles, there are at most  $|\mathcal{R}(Y)| - 1$  inner edges.

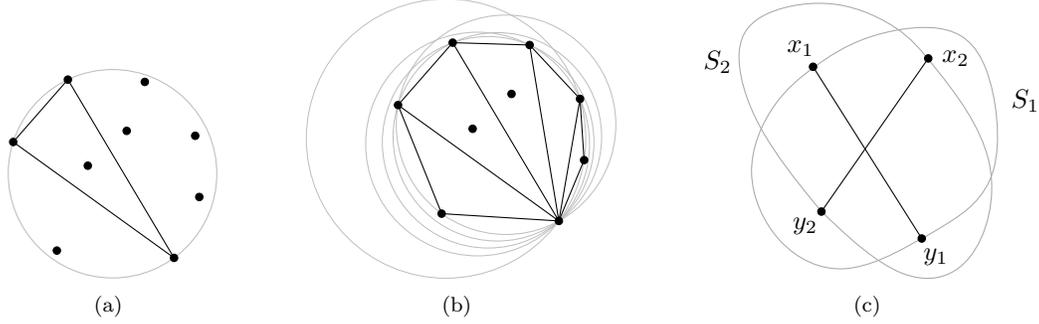


FIGURE 9. (a) A set  $Y$  of 9 points that is captured by an  $S$ -ranges, for  $S$  being a disc, and the corresponding triangle in  $G(Y)$ . (b) All  $S$ -ranges containing  $Y$  with at least two points of  $Y$  on their boundary and the entire graph  $G(Y)$ . (c) If two edges  $x_1y_1$  and  $x_2y_2$  in  $G(Y)$  cross, then the corresponding  $S$ -ranges have at least four intersections on their boundaries.

- (ii) Consider a representative range  $S_1$  for a hyperedge  $Y$  with  $\{p_1, q_1\} = Y \cap \partial S_1$ . Assume for the sake of contraction that there is another representative range,  $S_2$  for  $Y$  with  $\{p_2, q_2\} = Y \cap \partial S_2$  and  $\{p_2, q_2\} \neq \{p_1, q_1\}$ . We have that  $p_2, q_2 \in S_1, p_1, q_1 \in S_2$ . Assume, without loss of generality that  $q_2 \notin \{p_1, q_1\}$  and  $q_1 \notin \{p_2, q_2\}$ . Then  $q_2 \in S_1 - \partial S_1$  and  $q_1 \in S_2 - \partial S_2$ .

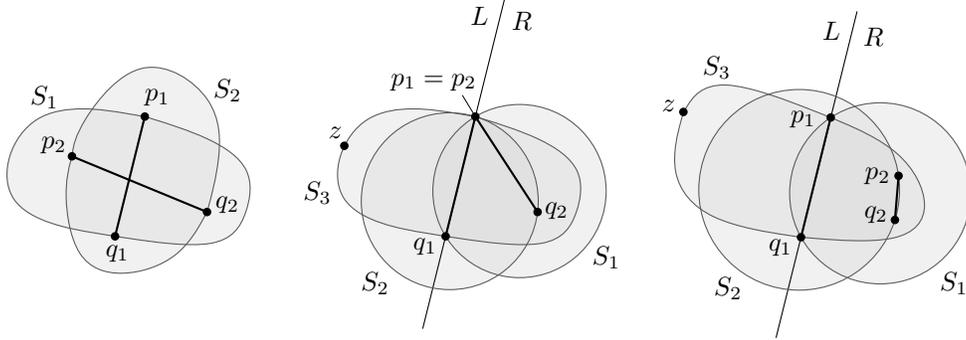


FIGURE 10. The three cases in the proof of Lemma 12 (ii).

We need to distinguish the following cases: segments  $p_2q_2$  and  $p_1q_1$  cross properly,  $p_2q_2$  and  $p_1q_1$  share a vertex, i.e.,  $p_2 = p_1$ , and finally  $p_1q_1$  is to the left of  $p_2q_2$ . The first case does not occur by the argument presented in the previous item. In the other two cases, let  $L$  and  $R$  be the halfplanes defined by  $\overline{p_1q_1}$  not containing  $q_2$  and containing  $q_2$ , respectively. Consider the  $S$ -range  $S_3 = \text{next}_L(S_1)$ , which exists by Lemma 8 as  $S_1 \Delta L$  contains  $q_2$ . We see that  $R$  contains at least one point in  $\partial S_3 \cap \partial S_2$ , see Figure 10. Moreover,  $S_3$  must contain a point  $z \in X \setminus Y$  because it is a next range and  $Y$  is not of Type I. It follows that  $z \in L \setminus S_2$ , which implies that the closure of  $L$  contains two points in  $\partial S_3 \cap \partial S_2$ , a contradiction to Lemma 6 (i).

□

KARLSRUHER INSTITUT FÜR TECHNOLOGIE, KARLSRUHE, GERMANY  
*E-mail address:* `maria.aksenovich@kit.edu`

KARLSRUHER INSTITUT FÜR TECHNOLOGIE, KARLSRUHE, GERMANY  
*E-mail address:* `torsten.ueckerdt@kit.edu`