

A note on saturation for Berge- G hypergraphs

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Abstract

For a graph $G = (V, E)$, a hypergraph H is called *Berge- G* if there is a hypergraph H' , isomorphic to H , so that $V(G) \subseteq V(H')$ and there is a bijection $\phi : E(G) \rightarrow E(H')$ such that for each $e \in E(G)$, $e \subseteq \phi(e)$. The set of all Berge- G hypergraphs is denoted $\mathcal{B}(G)$.

A hypergraph H is called *Berge- G saturated* if it does not contain any subhypergraph from $\mathcal{B}(G)$, but adding any new hyperedge of size at least 2 to H creates such a subhypergraph.

Since each Berge- G hypergraph contains $|E(G)|$ hyperedges, it follows that each Berge- G saturated hypergraph must have at least $|E(G)| - 1$ edges. We show that for each graph G that is not a certain star and for any $n \geq |V(G)|$, there are Berge- G saturated hypergraphs on n vertices and exactly $|E(G)| - 1$ hyperedges. This solves a problem of finding a saturated hypergraph with the smallest number of edges exactly.

1 Introduction

For a graph $G = (V, E)$, a hypergraph H is called *Berge- G* if there is a hypergraph H' , isomorphic to H , so that $V(G) \subseteq V(H')$ and there is a bijection $\phi : E(G) \rightarrow E(H')$ such that for each $e \in E(G)$, $e \subseteq \phi(e)$. The set of all Berge- G hypergraphs is denoted $\mathcal{B}(G)$.

Here, for a graph or a hypergraph F , we shall always denote the vertex set of F as $V(F)$ and the edge set of F as $E(F)$. A copy of a graph F in a graph G is a subgraph of G isomorphic to F . When clear from context, we shall drop the word “copy” and just say that there is an F in G .

Several classical questions regarding Berge- G hypergraphs have been considered. Among those are extremal numbers for Berge- G hypergraphs measuring the largest number of hyperedges or the largest weight of hypergraphs on n vertices that contain no subhypergraph from $\mathcal{B}(G)$, see for example [5, 9, 6, 11]. In addition, Ramsey numbers for Berge- G hypergraphs have been considered in [1, 8, 7].

In this paper, we consider a saturation problem. Let \mathcal{F} be a class of hypergraphs with edges of size at least two. A hypergraph \mathcal{H} is called *\mathcal{F} saturated* if it does not contain any subhypergraph isomorphic to a member of \mathcal{F} , but adding any new hyperedge of size at least 2 to \mathcal{H} creates such a subhypergraph.

Saturation problem for families of k -uniform hypergraphs has been treated by Pikhurko [12], see also [13]. Pikhurko [12] proved in particular, that for any k -uniform hypergraph G there is an n -vertex k -uniform hypergraph H that is $\{G\}$ saturated and has $O(n^{k-1})$

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edges. This extends a result of Kászonyi and Tuza [10] who proved this fact for $k = 2$, i.e., for graphs. See also a survey of Faudree et al. [4]. Here $\{G\}$ saturated means that H has no subhypergraph isomorphic to G but adding any new hyperedge of size k creates such a subhypergraph. This result is asymptotically tight for some G . The determination of a smallest size for $\{G\}$ -saturated hypergraphs remains open in general. In the same setting of k -uniform hypergraphs, English et al. [2] proved that there are $\mathcal{B}_k(G)$ saturated hypergraphs on n vertices and $O(n)$ hyperedges, where $\mathcal{B}_k(G)$ is the set of all k -uniform Berge- G hypergraphs, $3 \leq k \leq 5$. See also English et al. [3], for Berge saturation results on some special graphs.

We restrict our attention to the non-uniform case and Berge- G hypergraphs. For $n \geq |V(G)|$, let the *saturation number* for a Berge- G hypergraph be defined as

$$\text{sat}(n, \mathcal{B}(G)) = \min\{|E(\mathcal{H})| : \mathcal{H} \text{ is a } \mathcal{B}(G) \text{ saturated hypergraph on } n \text{ vertices}\}.$$

Observe that for any nontrivial graph G ,

$$\text{sat}(n, \mathcal{B}(G)) \geq |E(G)| - 1.$$

Since no Berge- G hypergraph has hyperedges of sizes less than 2, we can assume that all hypergraphs considered have hyperedges of sizes at least 2. We further assume that graphs considered have no isolated vertices. The following is the main result of this paper:

Theorem 1. *Let $G = (V, E)$ be a graph with no isolated vertices, $n \geq |V(G)|$, and $m = |E(G)| - 1$. Then*

$$\text{sat}(n, \mathcal{B}(G)) = \begin{cases} |E(G)|, & \text{if } G \text{ is a star on at least four edges,} \\ |E(G)| - 1, & \text{otherwise.} \end{cases}$$

Moreover if G_1 is a star on at least 4 edges and G_2 is any other graph, then $\mathcal{H}_t(n)$ and $\mathcal{H}(n, m)$ are a Berge- G_1 and a Berge- G_2 saturated hypergraphs, respectively.

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$. We shorten $\{i, j\}$ as ij when clear from context. If F is a hypergraph and e is a hyperedge, we denote by $F + e$, $F - e$, a hypergraph obtained by adding e to F , deleting e from F , respectively.

Construction of a hypergraph $\mathcal{H}_t(n)$:

Let n and t be positive integers, $t \leq n$. Let $\mathcal{H}_t(n) = ([n], \{[n], [n] - \{1\}, [n] - \{2\}, \dots, [n] - \{t-3\}, [n] - \{t-2\}\})$.

Construction of a set system $H'(n, m)$ and a hypergraph $\mathcal{H}(n, m)$:

Let n and m be positive integers, $m \leq \binom{n}{2}$. Let $x = \min\{m-1, n\}$. Let V' be a set of singletons, $V' \subseteq \{\{i\} : i \in [n]\}$, $|V'| = x$. Let E' be an edge-set of an almost regular graph (the degrees of vertices differ by at most one) on the vertex set $[n]$, such that $|E'| = m - x - 1$. Let $H'(n, m) = \{\emptyset\} \cup V' \cup E'$.

Informally, we build a set system $H'(n, m)$ of m sets on the ground set $[n]$ by first picking an empty set, then as many as possible singletons, and then pairs, so that the pairs form an edge-set of an almost regular graph.

Let

$$\mathcal{H} = \mathcal{H}(n, m) = ([n], \{[n] - E : E \in H'(n, m)\}).$$

Note that $|E(\mathcal{H})| = m$ and each hyperedge of \mathcal{H} has size n , $n - 1$, or $n - 2$.

Examples.

If $n = 4$ and $m = 4$, we have:

$$\begin{aligned} H'(4, 4) &= \{\emptyset, \{1\}, \{2\}, \{3\}\}, \\ E(\mathcal{H}(4, 4)) &= \{[4], \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}\}. \end{aligned}$$

If $n = 5$ and $m = 8$, we have

$$H'(5, 8) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{12\}, \{34\}\},$$

$$E(\mathcal{H}(5, 8)) = \{[5], \{2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 3, 4\}, \{3, 4, 5\}, \{1, 2, 5\}\}.$$

Let H be a Berge- G hypergraph, we call a copy G' of G , where $V(G') \subseteq V(H)$ and the edges of G' are contained in distinct hyperedges of H , an *underlying graph* of the Berge- G hypergraph H . For example, if G' is a triangle on vertices $1, 2, 3$, then a hypergraph $(\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\})$ is Berge- K_3 and G' is an underlying graph of Berge- K_3 hypergraph H .

2 Proof of the main theorem

Let S_t denote a star on t vertices.

Lemma 2. *Let $t \geq 5$, $n \geq t$. Then $\text{sat}(n, \mathcal{B}(S_t)) = t - 1 = |E(S_t)|$.*

Proof. To show the lower bound, assume first that there is a hypergraph \mathcal{H} on $t - 2$ hyperedges and vertex set $[n]$ that is Berge- S_t saturated. Since maximum degree of any member in $\mathcal{B}(S_t)$ is at least $t - 1$, we have that the maximum degree of $\mathcal{H} + e$ for any new edge e of size at least 2 is at least $t - 1$. We have that \mathcal{H} has at least $|V(S_t)| = t \geq 5$ vertices. Assume first that \mathcal{H} contains an edge of size 2, say 12 . Then any vertex in $\{3, \dots, n\}$ does not belong to this edge, so it has a degree at most $t - 3$. Thus, for any $i, j \in \{3, \dots, n\}$, $i \neq j$, the maximum degree of $\mathcal{H} + ij$ is at most $t - 2$, implying that $ij \in E(\mathcal{H})$. Since the edge 12 was chosen arbitrarily, we can conclude that \mathcal{H} contains all edges of size 2. Thus \mathcal{H} has at least $\binom{n}{2} \geq \binom{t}{2} > t - 2$ edges, a contradiction. Therefore \mathcal{H} has no hyperedges of size 2. Assume next that all but at most one vertex, say n , belong to all hyperedges of \mathcal{H} . Thus each hyperedge contains the set $[n - 1]$, implying that each hyperedge is either $[n - 1]$ or $[n]$, a contradiction to the fact that there are $t - 2 \geq 3$ distinct hyperedges in $E(\mathcal{H})$. Hence, there are two vertices, say 1 and 2 , each with degree at most $t - 3$. We know that $12 \notin E(\mathcal{H})$ and that $\mathcal{H} + 12$ has maximum degree at most $t - 2$, a contradiction. Thus \mathcal{H} is not $\mathcal{B}(S_t)$ saturated.

For the upper bound, we show that \mathcal{H}_t is a $\mathcal{B}(S_t)$ -saturated hypergraph. Recall that $\mathcal{H} = \mathcal{H}_t = ([n], \{[n], [n] - \{1\}, [n] - \{2\}, \dots, [n] - \{t - 3\}, [t - 3]\})$.

Note that each vertex of \mathcal{H} has degree $t - 2$. Thus \mathcal{H} is $\mathcal{B}(S_t)$ -free. Let $e \subseteq [n]$ of size at least 2, such that $e \notin E(\mathcal{H})$. Let $i, j \in e$, $i \neq j$. We shall show that $\mathcal{H} + e$ contains a Berge S_t hypergraph.

Case 1. $i, j \in [t - 3]$, without loss of generality $i = 1, j = 2$. Then the pairs $1n, 1(n - 1), 13, \dots, 1(t - 2), 12$ are contained in $[n], [n] - \{2\}, \dots, [n] - \{t - 3\}, [t - 3], e$, respectively, and form an underlying graph of Berge- S_t in $\mathcal{H} + e$.

Case 2. i or j is not in $[t-3]$. Let, without loss of generality $i = n$. Then, without loss of generality $j = n-1$ or $j = 1$. Then the pairs $n2, n3, \dots, n(t-2)$ are contained in $[n] - \{1\}, [n] - \{2\}, \dots, [n] - \{t-3\}$, respectively, and the pairs $1n, (n-1)n$ are contained in $[n], e$ or $e, [n]$, respectively. Thus all these $t-1$ pairs form an underlying graph of Berge- S_t in $\mathcal{H} + e$. \square

Proof of Theorem 1. First we consider some special graphs: stars on at most three edges and a triangle. For the upper bounds on $\text{sat}(n, \mathcal{B}(G))$ for $G = S_2, S_3, S_4, K_3$, consider the following hypergraphs in order for $n \geq 2, n \geq 3, n \geq 4$, and $n \geq 3$, respectively: $([n], \emptyset), ([n], \{[n]\}), ([n], \{[n], [n] - \{1\}\}), ([n], \{[n], [n] - \{1\}\})$. It is easy to see that these hypergraphs are saturated for the respective Berge hypergraphs. Thus, for G being one of these graphs, $\text{sat}(\mathcal{B}(G)) \leq |E(G)| - 1$. Since the lower bound on $\text{sat}(\mathcal{B}(G))$ is trivially $|E(G)| - 1$, the theorem holds in this case. Lemma 2 implies that the theorem holds for all other stars.

From now on, let G be a non-empty graph which is neither a star nor a K_3 . Let n be the number of vertices in G , $n \geq 4$. We shall further assume that G has no isolated vertices and that $V(G) = [n]$. Let $m = |E(G)| - 1$. We shall prove that $\mathcal{H} = \mathcal{H}(n, m)$ as defined in the introduction is a Berge- G saturated hypergraph, i.e. such that it does not contain any member of $\mathcal{B}(G)$ as a subhypergraph and such that for any new hyperedge e of size at least two, $\mathcal{H} + e$ contains a Berge- G sub-hypergraph. In fact, instead of $\mathcal{H}(n, m)$ we shall be mostly using the system $H'(n, m)$ also defined in the introduction. Note that \mathcal{H} does not contain any member of $\mathcal{B}(G)$ since \mathcal{H} has $|E(G)| - 1$ edges.

Consider $e, e \subseteq [n], |e| \geq 2, e \notin E(H)$. Let $\{i, j\} \subseteq e, i \neq j$. Relabel the vertices of G such that $ij \in E(G)$ and i is a vertex of maximum degree in G . We shall show that \mathcal{H} is a Berge- $(G - ij)$, thus showing that $\mathcal{H} + e$ is Berge- G . We shall prove one of the following equivalent statements:

- (i) there is a bijection ϕ between $E(G - ij)$ and $E(\mathcal{H})$ such that $e' \subseteq \phi(e')$ for any $e' \in E(G - ij)$,
- (ii) there is a bijection f between $E(G - ij)$ and $H' = H'(n, m)$ such that for each $e' \in E(G - ij)$, $e' \cap f(e') = \emptyset$,
- (iii) there is a perfect matching in a bipartite graph F with one part $A = E(G) - \{ij\}$ and the other part $B = H'$ such that $e' \in A = E(G) - \{ij\}$ and $e'' \in B = H'$ are adjacent in F iff $e' \cap e'' = \emptyset$.

One can see that (i) and (ii) are equivalent by defining $\phi(e')$ to be $[n] - f(e')$. The equivalence of (ii) and (iii) is clear since $|A| = |B|$. Next, we shall prove (iii).

In each of the cases below, we assume that there is no perfect matching in F , thus by Hall's theorem, there is a set $S \subseteq A$ such that $|N_F(S)| < |S|$. Let $Q = B \setminus N_F(S)$. We see that each element of Q intersects each edge in S . Let G_S be a subgraph of G with edge set S . Since each element in Q has size one or two, G_S has a vertex cover of size one or two. Thus G_S is either a star, a triangle, or an edge-disjoint union of two stars. Clearly, \emptyset is not in Q . Assume some singleton, say $\{1\}$ is in Q . Then S forms a star with center 1. Then all singletons $\{2\}, \{3\}, \dots$ and \emptyset are in $N_F(S)$. If $S \neq A$, i.e., $|E(G)| - 1 > |S|$, then $|N_F(S)| \geq |S|$, a contradiction to our assumption on S . If $S = A$, i.e., G is a union

of a star and an edge ij , since i is a vertex of maximum degree in G , we see that G is a star, a contradiction. Thus we can assume that Q contains only two-element sets, i.e., in particular H' has two-element sets and thus, by definition of H' , $|H'| > n + 1$. Finally, since an empty set and all singletons are not in Q , they are in $N_F(S)$, so $|N_F(S)| \geq n$. Thus $|S| \geq n + 1$, and in particular, S does not form a star. We observed earlier that we could assume that G is not a star.

Case 1. G is a union of two stars.

We already excluded the case that G is a star, so let's assume that G is an edge-disjoint union of two stars with different centers. If one of the stars has at most two edges, then $|E(G)| \leq n + 1$, and $|S| \leq n$, a contradiction. Thus each of the stars has at least 3 edges.

Note that G has at most $2n - 1$ edges. In particular, since there are n singletons and an empty set in H' and $|H'| \leq 2n - 1$, we have that E' , the set of pairs from H' , has size at most $n - 2$ and thus the graph on edge set E' has maximum degree at most 2. This implies that for every vertex there is a non-adjacent vertex in a graph with edge-set E' . Let k be the integer such that $ik \notin E'$. Relabel the vertices of G such that ij is an edge of G , and i and k are the centers of the stars whose union is G , and $j \neq k$. Since $ik \notin E'$, it follows that $ik \notin Q$. Since each pair from Q forms a vertex cover of G_S , there is a pair different from ik that forms a vertex cover of G_S . Since ik is a vertex cover of G , it is a vertex cover of G_S . Thus G_S has two distinct vertex covers of size 2. Then G_S is a subgraph of a triangle with possibly some further edges incident to the same vertex of the triangle or a subgraph of a C_4 . This implies that $|S| \leq n$, a contradiction.

Case 2. G is not a union of two stars.

If $|Q| = 1$, then $|N_F(S)| = |B| - 1 = |A| - 1$. Since $|S| > |N_F(S)| = |A| - 1$ and $S \subseteq A$, we have that $S = A$, hence $G_S = G - ij$. Since there is a vertex cover of G_S of size 2, we have that $G_S = G - ij$ is a union of two stars S', S'' , so G is a union of two stars and an edge incident to a vertex of maximum degree of G . If maximum degree of G is at least four, then i is a center of S' and S'' . Thus G is a union of two stars, a contradiction. If the maximum degree of G is at most 3, then $|E(G)| \leq 7$. On the other hand, $m = |H'| \geq n + 2$. Thus $n + 3 \leq |E(G)| \leq 7$. Thus $n = |V(G)| \leq 4$ and for each such choice of n we reach a contradiction by the fact that $n + 3 \leq |E(G)|$. If Q contains two disjoint edges, say 12 and 34, then G_S can only be a subgraph of a 4-cycle 13241. So, $|S| \leq 4 \leq n$, a contradiction to our assumption that $|S| \geq n + 1$.

Thus Q contains edges that either form a star on at least three edges or a subgraph of a triangle. If the edges of Q form a star on at least three edges, say 12, 13, 14, \dots , S forms a star with center 1, a contradiction. If the edges of Q form a triangle, say 123, then we arrive at a contradiction since no two-element set can at the same time intersect 12, 23, and 13. Thus Q contains exactly two adjacent edges, say 12 and 13. It follows that S forms a star with center 1 and maybe an edge 23. Then $|S| \leq n$, a contradiction. Hence, there is a perfect matching in F and thus \mathcal{H} is Berge- G saturated. \square

3 Conclusions

In this note, we completely determine $\text{sat}(n, \mathcal{B}(G))$ for any $n \geq |V(G)|$ and show in particular that this function does not depend on n . There are many variations of saturation numbers for non-uniform hypergraphs that could be considered. Among those are functions

optimising the total weight of a saturated hypergraph, i.e., the sum of cardinalities of all hyperedges, or functions optimising the size of a saturated multihypergraph. These have been considered by the second author in [14]. One particularly interesting variation considered in [14] is the following notion of saturation: a hypergraph \mathcal{H} is called strongly \mathcal{F} saturated with respect to a family of hypergraphs \mathcal{F} if \mathcal{H} does not contain any member of \mathcal{F} as a subhypergraph, but replacing any hyperedge e of \mathcal{H} with $e \cup \{v\}$ for any vertex $v \notin e$ such that $e \cup \{v\} \notin E(\mathcal{H})$ creates such a member of \mathcal{F} .

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