

Chromatic number of ordered graphs with forbidden ordered subgraphs.

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Abstract

It is well-known that the graphs not containing a given graph H as a subgraph have bounded chromatic number if and only if H is acyclic. Here we consider *ordered graphs*, i.e., graphs with a linear ordering \prec on their vertex set, and the function

$$f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\},$$

where $\text{Forb}_{\prec}(H)$ denotes the set of all ordered graphs that do not contain a copy of H .

If H contains a cycle, then as in the case of unordered graphs, $f_{\prec}(H) = \infty$. However, in contrast to the unordered graphs, we describe an infinite family of ordered forests H with $f_{\prec}(H) = \infty$. An ordered graph is crossing if there are two edges uv and $u'v'$ with $u \prec u' \prec v \prec v'$. For connected crossing ordered graphs H we reduce the problem of determining whether $f_{\prec}(H) \neq \infty$ to a family of so-called monotonically alternating trees. For non-crossing H we prove that $f_{\prec}(H) \neq \infty$ if and only if H is acyclic and does not contain a copy of any of the five special ordered forests on four or five vertices, which we call bonnets. For such forests H , we show that $f_{\prec}(H) \leq 2^{|V(H)|}$ and that $f_{\prec}(H) \leq 2|V(H)| - 3$ if H is connected.

1 Introduction

What conclusions can one make about the chromatic number of a graph knowing that it does not contain certain subgraphs? Let H be a graph on at least two vertices, $\text{Forb}(H)$ be the set of all graphs not containing H as a subgraph, and $f(H) = \sup\{\chi(G) \mid G \in \text{Forb}(H)\}$. If H has a cycle of length ℓ , then for any integer χ there is a graph G of girth at least $\ell+1$ and chromatic number χ , see [11], implying that $f(H) = \infty$. On the other hand, if H is a forest on k vertices and G is a graph of chromatic number at least k , then G contains a k -critical subgraph G' , that in turn has minimum degree at least $k-1$. Thus a copy of H can be found as a subgraph of G' by a greedy embedding. Therefore $G \notin \text{Forb}(H)$, implying that $f(H) \leq k-1$. So, we see that $f(H)$ is finite if and only if H is acyclic.

A similar situation holds for directed graphs, with a similarly defined function $f_{\text{dir}}(H)$ being finite if and only if the underlying graph of H is acyclic. A result of Addalirio-Berry *et al.* [1], see also [4], implies that $f_{\text{dir}}(H) \leq k^2/2 - k/2 - 1$ whenever H is a directed k -vertex graph whose underlying graph is acyclic.

Here, we consider the behavior of the chromatic number of ordered graphs with forbidden ordered subgraphs. An *ordered graph* G is a graph (V, E) together with a linear ordering \prec of its vertex set V . An *ordered subgraph* H of an ordered graph G is a subgraph of the (unordered) graph (V, E) together with the linear ordering of its vertices inherited from G . An ordered subgraph H is a *copy of an ordered graph* H' if there is an order preserving isomorphism between H and H' . For an ordered graph H on at least two vertices¹ let $\text{Forb}_{\prec}(H)$ denote the set of all ordered graphs that do not contain a copy of H . We consider the function f_{\prec} given by

$$f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\}.$$

We show that it is no longer true that $f_{\prec}(H)$ is finite if and only if H is acyclic. When H is connected, we reduce the problem of determining whether $f_{\prec}(H) \neq \infty$ to a well behaved class of trees, which we call monotonically alternating trees. We completely classify so-called “non-crossing” ordered graphs H for which $f_{\prec}(H) = \infty$. In case of “non-crossing” H with finite $f_{\prec}(H)$, we provide specific upper bounds on this function in terms of the number of vertices in H . Note that $f_{\prec}(H) \geq |V(H)| - 1$ for any ordered graph H , since a complete graph on $|V(H)| - 1$ vertices is in $\text{Forb}_{\prec}(H)$.

We need some formal definitions before stating the main results of the paper. We consider the vertices of an ordered graph laid out along a horizontal line according to their ordering \prec and say that for $u \prec v$ the vertex u is to the *left of* v and the vertex v is to the *right of* u . We write $u \preceq v$ if $u \prec v$ or $u = v$. For two sets of vertices U and U' we write $U \prec U'$ if all vertices in U are left of all vertices in U' . Two edges uv and $u'v'$ **cross** if $u \prec u' \prec v \prec v'$ and an ordered graph H is called *crossing* if it contains two crossing edges. Otherwise, H is called *non-crossing*. Two distinct ordered subgraphs G_1 and G_2 of the same ordered graph H *cross each other* if there is an edge in G_1 crossing an edge in G_2 .

An ordered graph is a **bonnet** if it has 4 or 5 vertices $u_1 \prec u_2 \preceq u_3 \prec u_4 \preceq u_5$ and edges u_1u_2, u_1u_5, u_3u_4 , or if it has vertices $u_1 \preceq u_2 \prec u_3 \preceq u_4 \prec u_5$ and edges u_1u_5, u_4u_5, u_2u_3 . See Figure 1 (first two rows). An ordered path $P = u_1, \dots, u_n$ is a **tangled path** if for a vertex u_i , $1 < i < n$, that is either leftmost or rightmost in P there is an edge in the subpath u_1, \dots, u_i that crosses an edge in the subpath u_i, \dots, u_n . See Figure 1 (last row, left and middle). Note that there are crossing paths which are not tangled, see for example Figure 1 (right).

¹If H has only one vertex, then $\text{Forb}_{\prec}(H)$ consists only of the graph with empty vertex set and one can think of $f_{\prec}(H)$ as being equal to 0. However, we will avoid this pathologic case throughout.

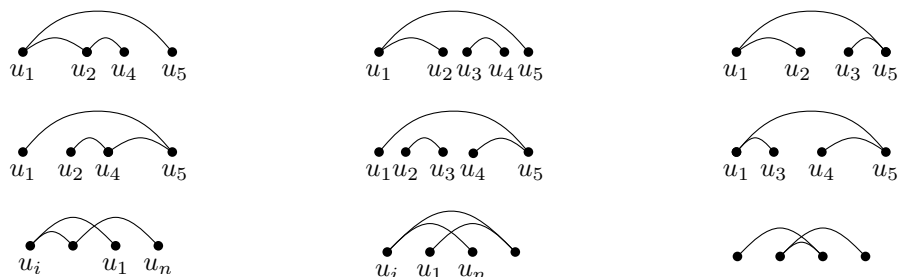


Figure 1: All bonnets (first two rows), two tangled paths (last row, left and middle) and a crossing path that is not tangled (last row, right).

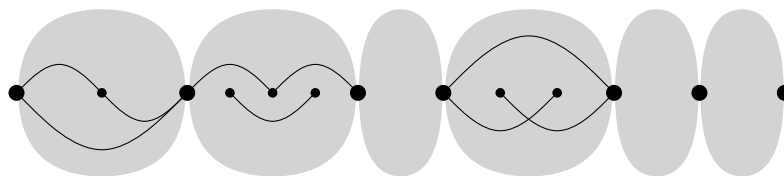


Figure 2: Segments of an ordered graph. The bold vertices are either inner cut-vertices or left-, rightmost vertices.

Theorem 1. *If an ordered graph H contains a cycle, a bonnet, or a tangled path, then $f_{\prec}(H) = \infty$.*

A vertex v of an ordered graph G is called an *inner cut vertex* if there is no edge uw with $u \prec v \prec w$ in G and v is not leftmost or rightmost in G . An *interval* in an ordered graph G is a set I of vertices such that for all vertices $u, v \in I$, $x \in V(G)$ with $u \prec x \prec v$ we have $x \in I$. A **segment** of an ordered graph G with $|V(G)| \geq 2$ is an induced subgraph H of G such that $|V(H)| \geq 2$, $V(H)$ is an interval in G , the leftmost and rightmost vertices in H are either inner cut vertices of G or leftmost respectively rightmost in G , and all other vertices in H are not inner cut vertices in G . So, G is the union of its segments, any two segments share at most one vertex and the inner cut vertices of G are precisely the vertices contained in two segments of G . In particular, the number of inner cut vertices of G is exactly one less than the number of its segments. See Figure 2.

The *length* of an edge xy is the number of vertices v such that $x \prec v \prec y$. A shortest edge among all the edges incident to a vertex x is referred to as a *shortest edge incident to x* . Note that there is either 1 or 2 shortest edges incident to a given vertex in a connected graph on at least two vertices. Let U be a vertex set in an ordered tree T , such that each vertex in U has exactly one shortest edge incident to it. For such a set U , let $S(U)$ be the set of edges e_u such that e_u is a shortest edge incident to u , $u \in U$. We call an ordered tree T **monotonically alternating** if there is a partition $V(T) = L \dot{\cup} R$, with $L \prec R$, such that L and R are independent sets in T , $E = S(L) \cup S(R)$, and neither $S(L)$ nor $S(R)$ contains a pair of crossing edges.

Theorem 2. *An ordered tree T contains neither a bonnet nor a tangled path if and*

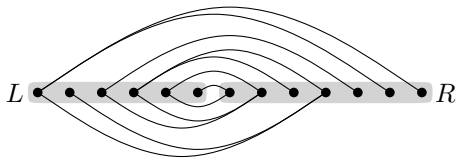


Figure 3: A monotonically alternating tree. Each edge on top is the shortest edge incident to a vertex in R and each edge at the bottom is the shortest edge incident to a vertex in L .

only if each segment of T is monotonically alternating. In particular if $f_{\prec}(H) \neq \infty$ for some connected ordered graph H , then each segment in H is a monotonically alternating tree.

Recall that an ordered graph is non-crossing if it does not contain any crossing edges. Note that a non-crossing graph does not contain tangled paths.

Theorem 3. *Let T be a non-crossing ordered graph on k vertices, $k \geq 2$. Then $f_{\prec}(T) \neq \infty$ if and only if T is a forest that does not contain a bonnet.*

Moreover, if $f_{\prec}(T) \neq \infty$ then $k-1 \leq f_{\prec}(T) \leq 2^k$. If, in addition T is connected, then $f_{\prec}(T) \leq 2k-3$. Finally, for each $k \geq 4$ there is an ordered non-crossing tree T with $k \leq f_{\prec}(T) \neq \infty$, while for $k=2,3$ we have $f_{\prec}(T) = k-1$.

For certain classes of ordered forests we prove better upper bounds on f_{\prec} . A k -nesting is an ordered graph T on vertices $u_1 \prec \dots \prec u_k \prec v_k \prec \dots \prec v_1$ and edges $u_i v_i$, $1 \leq i \leq k$. A k -crossing is an ordered graph T on vertices $u_1 \prec \dots \prec u_k \prec v_1 \prec \dots \prec v_k$ and edges $u_i v_i$, $1 \leq i \leq k$. We may omit the parameter k if it is not important. A *generalized star* is a union of a star and isolated vertices.

The following theorem summarizes several results on ordered forests which are either not covered by Theorem 3 or improve the upper bound from Theorem 3 significantly.

One of the known classes of such graphs is a special family of star forests, or, in other words, tuple matchings. For positive integers m and t and a permutation π of $[t]$, an m -tuple t -matching $M = M(t, m, \pi)$ is an ordered graph with vertices $v_1 \prec \dots \prec v_{t(m+1)}$, where each edge is of the form $v_i v_{t+j+m(\pi(i)-1)}$ for $1 \leq i \leq t$, $1 \leq j \leq m$. I.e., an m -tuple t -matching is a vertex disjoint union of t stars on m edges each, where v_1, \dots, v_t are the centers of the stars that are to the left of all leaves and the leaves of each star form an interval in M , so that these intervals are ordered according to the permutation π . The third item in the following theorem is an immediate corollary of a result by Weidert [19] who provides a linear upper bound on the extremal function for M . The other results are based on linear upper bounds for the extremal functions of nestings due to Dujmovic and Wood [10], on the extremal function of crossings due to Capocyleas and Pach [5] and lower bounds for ordered Ramsey numbers due to Conlon *et al.* [7], see also Balko *et al.* [2]. See Section 3 for a more detailed description of extremal functions and ordered Ramsey numbers.

Theorem 4. *Let T be an ordered forest on k vertices, $k \geq 2$.*

- *If each segment of T is either a generalized star, a 2-nesting, or a 2-crossing, then $f_{\prec}(T) = k - 1$.*
- *If each segment of T is either a nesting, a crossing, a generalized star, or a non-crossing tree without bonnets, then $k - 1 \leq f_{\prec}(T) \leq 2k - 3$.*
- *If T is an m -tuple t -matching for some positive integers m and t , then $k - 1 \leq f_{\prec}(T) \leq 2^{10k \log(k)}$.*
- *There is a positive constant c such that for each even positive integer $k \geq 4$ there is a matching M on k vertices with $f_{\prec}(M) \geq 2^{c \frac{\log(k)^2}{\log \log(k)}}$.*

The paper is organized as follows. In Section 2 we introduce all missing necessary notions. In Section 3 we summarize the known results on extremal functions and Ramsey numbers for ordered graphs and show how they could be used in determining f_{\prec} . In Section 4 we prove some structural lemmas and provide several reductions that are used in the proofs of the main results and that might be of independent interest. Section 5 contains the proofs of Theorems 1–4. We summarize all known results for forests with at most three edges in Section 6. Finally, Section 7 contains conclusions and open questions.

2 Definitions

Let K_n denote a complete graph on n vertices. For a positive integer n and an ordered graph H , let $\text{ex}_{\prec}(n, H)$ denote the *ordered extremal number*, i.e., the largest number of edges in an ordered graph on n vertices in $\text{Forb}_{\prec}(H)$. For an ordered graph H the *ordered Ramsey number* $R_{\prec}(H)$ is the smallest integer n such that in any edge-coloring of an ordered K_n in two colors there is a monochromatic copy of H . Recall that an interval in an ordered graph G is a set I of vertices such that for all vertices $u, v \in I$, $x \in V(G)$ with $u \prec x \prec v$ we have $x \in I$. The *interval chromatic number* $\chi_{\prec}(G)$ of an ordered graph G is the smallest number of intervals, each inducing an independent set in G , needed to partition $V(G)$. An inner cut vertex v of an ordered graph G *splits* G into ordered graphs G_1 and G_2 if G_1 is induced by all vertices u with $u \preceq v$ in G and G_2 is induced by all vertices u with $v \preceq u$. A vertex of degree 1 is called a *leaf*. A vertex in an ordered graph G is called *reducible*, if it is a leaf in G , is leftmost or rightmost in G and has a common neighbor with the vertex next to it. We call an edge uv in a graph G *isolated* if u and v are leaves in G . A graph G is *t -degenerate* if each subgraph of G has a vertex of degree at most t . A vertex v is *between* vertices u and w if $u \preceq v \preceq w$. The *reverse* \overline{G} of an ordered graph G is the ordered graph obtained by reversing the ordering of the vertices in G . A *u - v -path* P is a path starting with u and ending with v , i.e., a path v_1, \dots, v_k with $u = v_1$, $v = v_k$. Given a path $P = v_1, \dots, v_k$ let $v_i P = v_i, \dots, v_k$ and $P v_i = v_1, \dots, v_i$. Similarly for a neighbor $v \notin V(P)$ of v_1 let

$vP = v, v_1, \dots, v_k$. If $U \subseteq V(G)$, $F \subseteq E(G)$ let $G[U]$, $G - U$ and $G - F$ denote the graphs $(U, E(G) \cap \binom{U}{2})$, $(V(G) \setminus U, E(G) \cap \binom{V(G) - U}{2})$, and $(V(G), E(G) \setminus F)$, respectively. In particular if $u, v \in V(G)$ then $G - \{u, v\}$ is the graph obtained by removing u and v from G , not the edge uv only. If $u \in V(G)$ let $G - u = G - \{u\}$. The definitions of tangled paths, bonnets, crossing edges and subgraphs, intervals, segments, inner cut-vertices, and monotonically alternating trees are given before the statements of the main theorems in the introduction. We shall typically denote a general ordered graph by H , a tree or a forest by T , and a larger ordered graph by G . For all other undefined graph theoretic notions we refer the reader to West [20].

3 Connections to known results

There are connections between the extremal number $\text{ex}_{\prec}(n, H)$ and the function $f_{\prec}(H)$. If there is a constant c such that $\text{ex}_{\prec}(n, H) < cn$ for every n , then

$$f_{\prec}(H) \leq 2c, \quad (1)$$

so $f_{\prec}(H)$ is finite. Indeed, if $\text{ex}_{\prec}(n, H) < cn$ then any $G \in \text{Forb}_{\prec}(H)$ has less than $c|V(G)|$ edges, and hence has a vertex of degree less than $2c$. Moreover, if $G \in \text{Forb}_{\prec}(H)$, then each subgraph of G is in $\text{Forb}_{\prec}(H)$, so each subgraph has a vertex of degree less than $2c$, so G is $(2c - 1)$ -degenerate. Therefore $\chi(G) \leq 2c$.

Ordered extremal numbers are studied in detail in [17]. Recall that $\chi_{\prec}(G)$ is the smallest number of intervals, each inducing an independent set, needed to partition the vertices of an ordered graph G . Pach and Tardos [17] prove that for each ordered graph H

$$\text{ex}_{\prec}(n, H) = \left(1 - \frac{1}{\chi_{\prec}(H) - 1}\right) \binom{n}{2} + o(n^2).$$

For ordered graphs with interval chromatic number 2, Pach and Tardos find a tight relation between the ordered extremal number and pattern avoiding matrices. For an ordered graph H with $\chi_{\prec}(H) = 2$ let $A(H)$ denote the 0-1-matrix where the rows correspond to the vertices in the first color and the columns to the vertices in the second color of a proper interval coloring of H in 2 colors and let $A(H)_{u,v} = 1$ if and only if uv is an edge in H . A 0-1-matrix B *avoids* another 0-1-matrix A if there is no submatrix in B which becomes equal to A after replacing some ones with zeros. For a 0-1-matrix A let $\text{ex}(n, A)$ denote the largest number of ones in an $n \times n$ matrix avoiding A . In [17] it is shown that for each ordered graph H with $\chi_{\prec}(H) = 2$ there is a constant c such that $\text{ex}(\lfloor \frac{n}{2} \rfloor, A(H)) \leq \text{ex}_{\prec}(n, H) \leq c \text{ex}(n, A(H)) \log n$. Thus, when $\text{ex}(n, A(H))$ is linear in n , one can guarantee that $\text{ex}_{\prec}(n, H) = O(n \log n)$, but this is not enough to claim that $f_{\prec}(H) \neq \infty$.

In addition, we see that there is no direct connection between $f_{\prec}(H)$ and $\text{ex}_{\prec}(n, H)$ because there are dense ordered graphs avoiding H for some ordered graphs H with small $f_{\prec}(H)$. A specific example for such a graph H is an ordered path $u_1 u_2 u_3$, with $u_1 \prec u_2 \prec u_3$. One can see from Theorem 4 that $f_{\prec}(H) = 2$, but a complete bipartite ordered graph G with all vertices of one bipartition class to the left of

all other vertices does not contain H and has $|V(G)|^2/4$ edges. However, for some ordered graphs H with interval chromatic number 2, one can show that $\text{ex}_{\prec}(n, H)$ is linear. This in turn, implies that $f_{\prec}(H)$ is finite.

Some of the extensive research on forbidden binary matrices and extremal functions for ordered graphs can be found in [3, 12, 14, 15, 16].

There are also connections between the Ramsey numbers $R_{\prec}(H)$ for ordered graphs and the function $f_{\prec}(H)$. If the edges of K_n , $n = R_{\prec}(H) - 1$, are colored in two colors without monochromatic copies of H , then both color classes form ordered graphs G_1 and G_2 not containing H as an ordered subgraph. Then one of the G_i 's has chromatic number at least \sqrt{n} , since a product of proper colorings of G_1 and G_2 yields a proper coloring of K_n . Therefore $f_{\prec}(H) \geq \sqrt{R_{\prec}(H) - 1}$. Ordered Ramsey numbers were recently studied by Conlon *et al.* [7] and Balko *et al.* [2]. Other research on ordered graphs includes characterizations of classes of graphs by forbidden ordered subgraphs [8, 13] and the study of perfectly ordered graphs [6].

4 Structural Lemmas and Reductions

In this section we first analyze the structure of ordered trees without bonnets and tangled paths. This leads to a proof of Theorem 2 in Section 5. Afterwards we establish several cases when $f_{\prec}(H)$ can be upper bounded in terms of $f_{\prec}(H')$ for a subgraph H' of H . This allows us to reduce the problem of whether $f_{\prec}(H) \neq \infty$ to the problem of whether $f_{\prec}(H') \neq \infty$. These *reductions* are the crucial tools in the proof of Theorem 3 in Section 5.

Lemma 4.1. *Let T be an ordered tree that does not contain a tangled path and let $u \prec v \prec w$ be vertices in T . If uw is an edge in T , then all vertices of the path connecting u and v in T are between u and w .*

Proof. Let P be the path in T that starts with v and ends with the edge uw . Let ℓ denote the leftmost vertex in P . Assume for the sake of contradiction that $\ell \prec u$. Then the path $vP\ell$ contains neither u nor w and therefore crosses the edge uw . Hence the paths $P\ell$ and ℓP cross and P is tangled, a contradiction. Therefore $\ell = u$. Due to symmetric arguments w is the rightmost vertex in P . Hence all vertices in P are between u and w . \square

Lemma 4.2. *Let T be an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment. Deleting any leaf from T yields an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment.*

Proof. Let uv be an edge in T incident to a leaf u and let $T' = T - u$. Then clearly T' is an ordered tree that contains neither a bonnet nor a tangled path. For the sake of contradiction assume that T' has at least two segments and let x be an inner cut vertex in T' . Then $x \neq u, v$ and is between u and v in T , since x is not an inner cut vertex in T . By reversing T if necessary we may assume that $v \prec x \prec u$. Let P be the v - x -path in T' . All vertices in P are between v and u by Lemma 4.1 applied

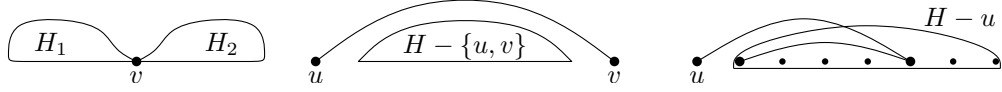


Figure 4: An inner cut vertex v splitting an ordered graph into ordered graphs H_1 and H_2 (left), an isolated edge uv in an ordered graph H (middle), and a reducible vertex u (right).

to u, v and x . In addition no vertex in P is to the right of x since x is an inner cut vertex in T' . So all vertices in P are between v and x . Let vw denote the first edge of P and let xy denote an edge in T' with $x \prec y$. Such an edge xy exists since the inner cut vertex x is not rightmost in T' and T' is connected. If $u \prec y$, then $uvPxy$ is a tangled path in T . If $y \prec u$, then u, v, w, x and y form a bonnet in T . In both cases we have a contradiction and hence T' has only one segment. \square

Lemma 4.3. *If T is an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment, then $\chi_{\prec}(T) \leq 2$.*

Proof. We prove the claim by induction on $k = |V(T)|$. If $k \leq 2$, then clearly $\chi_{\prec}(T) \leq 2$. So assume that $k \geq 3$. Let u denote a leaf in T , v its neighbor in T , and let $T' = T - u$. Then T' has only one segment and contains neither a bonnet nor a tangled path due to Lemma 4.2. Inductively $\chi_{\prec}(T') \leq 2$, i.e., there is a partition $L \dot{\cup} R = V(T')$, with $L \prec R$, such that all edges in T' are between L and R . By reversing T if necessary we assume that $v \in L$. For the sake of contradiction assume that $\chi_{\prec}(T) > 2$. Then $u \prec \ell$ for the rightmost vertex ℓ in L , possibly $\ell = v$. Let $w \in R$ denote one fixed neighbor of v in T' . Then all vertices of the path connecting ℓ and v in T' are between v and w due to Lemma 4.1. In particular ℓ is incident to an edge ℓx , $x \in R$, with $x \preceq w$. Hence $u \prec v$, since otherwise there is a bonnet on vertices v, u, ℓ, x , and w in T . If there is a vertex y , $u \prec y \prec v$, then all vertices of the path connecting y and u in T are between u and v due to Lemma 4.1. But this is not possible since $y, v \in L$ and all the neighbors of y are in R . Hence u is immediately to the left of v in T . Note that u is not leftmost in T , since otherwise v is an inner cut vertex in T . Consider the path P connecting a vertex left of u to ℓ in T . This path contains distinct vertices $p, q \in L$, $r \in R$, such that pr and rq are edges in P and $p \prec u \prec v \preceq q \prec r$. Hence there is a bonnet, a contradiction. This shows that $\chi_{\prec}(T) \leq 2$. \square

We now present several reductions. Let us mention that some of the following arguments are similar to reductions used for extremal numbers of matrices [17, 18].

Recall, that an inner cut vertex v of an ordered graph H splits H into ordered graphs H_1 and H_2 , where H_1 is induced by all vertices u with $u \preceq v$ in H and H_2 is induced by all vertices u with $v \preceq u$. See Figure 4 (left).

Reduction Lemma 1. *If an inner cut vertex v splits an ordered graph H into ordered graphs H_1 and H_2 with $f_{\prec}(H_1), f_{\prec}(H_2) \neq \infty$, then*

$$f_{\prec}(H) \leq f_{\prec}(H_1) + f_{\prec}(H_2).$$

Proof. Consider an ordered graph $G \in \text{Forb}_{\prec}(H)$. Let V_1 denote the set of vertices in G that are rightmost in some copy of H_1 in G . Further let $V_2 = V(G) \setminus V_1$. Then $G[V_2] \in \text{Forb}_{\prec}(H_1)$ by the choice of V_1 . Moreover $G[V_1] \in \text{Forb}_{\prec}(H_2)$, since otherwise the leftmost vertex u in a copy of H_2 in $G[V_1]$ is also a rightmost vertex in a copy of H_1 and hence plays the role of v in a copy of H in G . Thus $\chi(G) \leq \chi(G[V_1]) + \chi(G[V_2]) \leq f_{\prec}(H_2) + f_{\prec}(H_1)$ and since $G \in \text{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leq f_{\prec}(H_1) + f_{\prec}(H_2)$. \square

Reduction Lemma 2. *If v is an isolated vertex in an ordered graph H with $|V(H)| \geq 3$ and $f_{\prec}(H - v) \neq \infty$, then $f_{\prec}(H) \leq 2f_{\prec}(H - v)$.*

Proof. Consider an ordered graph $G \in \text{Forb}_{\prec}(H)$. If v is not leftmost or rightmost in H , then let V_1 be the set of vertices of G that are odd in the ordering of G and let $V_2 = V(G) \setminus V_1$. Then $G[V_1], G[V_2] \in \text{Forb}_{\prec}(H - v)$, since for any two vertices $u \prec w$ in V_i there is a vertex $v \in V_{3-i}$ with $u \prec v \prec w$, $i = 1, 2$. Hence $\chi(G) \leq \chi(G[V_1]) + \chi(G[V_2]) \leq 2f_{\prec}(H - v)$. If v is the leftmost or the rightmost in H , assume without loss of generality the former. Then clearly $G - u \in \text{Forb}_{\prec}(H - v)$ for the leftmost vertex u of G . Thus $\chi(G) \leq 1 + \chi(G - u) \leq 1 + f_{\prec}(H - v) \leq 2f_{\prec}(H - v)$. Since $G \in \text{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leq 2f_{\prec}(H - v)$ in both cases. \square

Reduction Lemma 3. *Let u and v be the leftmost and rightmost vertices in an ordered graph H , $|V(H)| \geq 4$. If uv is an isolated edge in H and $f_{\prec}(H - \{u, v\}) \neq \infty$, then*

$$f_{\prec}(H) \leq 2f_{\prec}(H - \{u, v\}) + 1.$$

Proof. See Figure 4 (middle). Let $H' = H - \{u, v\}$ and consider an ordered graph $G \in \text{Forb}_{\prec}(H)$. If G does not contain a copy of H' , then $\chi(G) \leq f_{\prec}(H') \leq 2f_{\prec}(H') + 1$. So, assume that G contains a copy of H' . Let $V_1 \dot{\cup} \dots \dot{\cup} V_p$ denote a partition of $V(G)$ into disjoint intervals with $V_1 \prec \dots \prec V_p$, v_i being the leftmost vertex in V_i , $1 \leq i \leq p$, such that $G[V_i] \in \text{Forb}_{\prec}(H')$, $1 \leq i \leq p$, and $G[V_i \cup \{v_{i+1}\}]$ contains a copy of H' , $1 \leq i < p$. Note that one can find such a partition greedily by iteratively choosing a largest interval from the left that does not induce any copy of H' in G . If $p \geq 3$, there are no edges xy with $x \in V_i$ and $v_{i+2} \prec y$, since otherwise xy together with a copy of H' in $G[V_{i+1} \cup \{v_{i+2}\}]$ forms a copy of H , $1 \leq i \leq p - 2$.

Choose a set Φ of $2f_{\prec}(H') + 1$ distinct colors. Let $\Phi_1, \dots, \Phi_p \subset \Phi$ denote subsets of colors such that $|\Phi_i| = f_{\prec}(H')$, $1 \leq i \leq p$, $\Phi_i \cap \Phi_{i+1} = \emptyset$, $1 \leq i < p$, and, if $p \geq 3$, $\Phi_{i+2} \setminus (\Phi_i \cup \Phi_{i+1}) \neq \emptyset$, $1 \leq i \leq p - 2$. Note that such sets Φ_i can be chosen greedily from Φ . Since $G[V_i] \in \text{Forb}_{\prec}(H')$ we can color $G[V_i]$ properly with colors from Φ_i , $1 \leq i \leq p$, such that, if $i \geq 3$, v_i is colored with a color in $\Phi_i \setminus (\Phi_{i-1} \cup \Phi_{i-2})$. This yields a proper coloring of G using colors from the set Φ only. Hence $\chi(G) \leq 2f_{\prec}(H') + 1$. Since $G \in \text{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leq 2f_{\prec}(H - \{u, v\}) + 1$. \square

Recall, that a vertex in an ordered graph H is called *reducible*, if it is a leaf in H , is leftmost or rightmost in H and has a common neighbor with the vertex next to it. See Figure 4 (right).

Reduction Lemma 4. *Let H denote an ordered graph with $|V(H)| \geq 3$. If u is a reducible vertex in H and $f_{\prec}(H - u) \neq \infty$, then*

$$f_{\prec}(H) \leq 2f_{\prec}(H - u).$$

Moreover, for each $G \in \text{Forb}_{\prec}(H)$ there is $G' \subseteq G$ such that G' is 1-degenerate and deleting the edges of G' from G yields a graph from $\text{Forb}_{\prec}(H - u)$.

Proof. By reversing H if necessary we may assume that the reducible vertex u is leftmost in H . Let $G \in \text{Forb}_{\prec}(H)$. Let E denote the set of edges in G consisting for each vertex w in G of the longest edge to the left incident to w in G , if such an edge exists.

Assume that there is a copy H' of $H - u$ in $G - E$. Let v denote the vertex in H' corresponding to the vertex immediately to the right of u in H and let w denote the vertex in H' corresponding to the neighbor of u in H . Then v is leftmost in H' and there is an edge between v and w in H' . Thus, there is an edge xw in E incident to w in G with $x \prec v$. Hence H' extends to a copy of H in G with the edge xw , a contradiction. This shows that $G - E \in \text{Forb}_{\prec}(H - u)$.

Finally observe that the graph G' with the edge-set E is 1-degenerate and hence 2-colorable. This shows that $\chi(G) \leq \chi(G')\chi(G - E) \leq 2f_{\prec}(H - u)$ and since $G \in \text{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leq 2f_{\prec}(H - u)$. \square

Having Reduction Lemma 4 at hand, we are now ready to prove that every non-crossing monotonically alternating tree T satisfies $f_{\prec}(T) \neq \infty$.

Lemma 4.4. *If T is a non-crossing monotonically alternating tree with $|V(T)| \geq 2$, then*

$$f_{\prec}(T) \leq 2|V(T)| - 3.$$

Proof. Let $k = |V(T)|$ and $G \in \text{Forb}_{\prec}(T)$. We shall prove that G can be edge-decomposed into $(k - 2)$ 1-degenerate graphs by induction on k .

If $k = 2$, then T consists of a single edge only. Hence G has an empty edge-set and there is nothing to prove.

So consider $k \geq 3$ and assume that the induction statement holds for all smaller values of k . Assume for the sake of contradiction that the leftmost vertex u and the rightmost w in T are of degree at least 2. Then the longest and the shortest edge incident to w do not coincide. Let e be the longest edge incident to w . Since in a monotonically alternating tree each edge is the shortest edge incident to its left or right endpoint, e is the shortest edge incident to its left endpoint. In particular, $e \neq uw$ because u is incident to another edge e' , shorter than uw . Thus e and e' cross since $\chi_{\prec}(T) \leq 2$, a contradiction. Hence the leftmost or the rightmost vertex is a leaf in T .

By reversing T if necessary we assume that u is of degree 1. We shall show that u is a reducible leaf. To do so, we need to show that the vertex x that is immediately to the right of u is adjacent to the neighbor v of u . Assume for the sake of contradiction that x is not adjacent to v . Note that v is adjacent to a leaf, so it

is not a leaf itself. Let e'' be an edge incident to v , $e'' \neq uv$. Then an edge incident to x crosses either uv or e'' since $\chi_{\prec}(T) \leq 2$, a contradiction. Thus x is adjacent to v and u is a reducible leaf in T .

Therefore, by Reduction Lemma 4, there is a 1-degenerate subgraph G' of G such that removing the edges of G' from G yields a graph $G'' \in \text{Forb}_{\prec}(T - u)$. Observe that the tree $T - u$ is non-crossing and monotonically alternating with $k > |V(T - u)| = k - 1 \geq 2$. Hence G'' can be edge-decomposed into $(k - 3)$ 1-degenerate graphs G_1, \dots, G_{k-3} by induction. Thus the graphs G_1, \dots, G_{k-3}, G' decompose G into $(k - 2)$ 1-degenerate graphs, proving the induction step.

If $k = 2$, we know that G has no edges and $\chi(G) = 1 \leq 2|V(T)| - 3$. So assume that $k \geq 3$. Since G is a union of $(k - 2)$ 1-degenerate graphs, each subgraph of G is a union of $(k - 2)$ 1-degenerate graphs, so each subgraph G^* of G on at least one vertex that has at most $(k - 2)(|V(G^*)| - 1)$ edges, and thus has a vertex of degree at most $2(k - 2) - 1$. Therefore G is $(2(k - 2) - 1)$ -degenerate, so $\chi(G) \leq 2(k - 2) \leq 2|V(T)| - 3$. Since $G \in \text{Forb}_{\prec}(H)$ was arbitrary we have $f_{\prec}(H) \leq 2|V(T)| - 3$. \square

Reduction Lemma 5. *Let T denote an ordered matching on at least 2 edges. If uv is an edge in T and u and v are consecutive and $f_{\prec}(T - \{u, v\}) \neq \infty$, then*

$$f_{\prec}(T) \leq 3 f_{\prec}(T - \{u, v\}).$$

Proof. Let $G \in \text{Forb}_{\prec}(T)$ with vertices $v_1 \prec \dots \prec v_n$. We shall prove that $\chi(G) \leq 3 f_{\prec}(T - \{u, v\})$ by induction on $n = |V(G)|$. If $n \leq 3 f_{\prec}(T - \{u, v\})$, then the claim holds trivially. So assume that $n > 3 f_{\prec}(T - \{u, v\}) \geq 3$. If there are two consecutive vertices x, y in G that are not adjacent, then let G' denote the graph obtained by identifying x and y . Then $G' \in \text{Forb}_{\prec}(T)$ and $\chi(G) \leq \chi(G')$. Hence $\chi(G) \leq \chi(G') \leq 3 f_{\prec}(T - \{u, v\})$ by induction. If each pair of consecutive vertices in G forms an edge, then consider a partition $V(G) = V_0 \cup V_1 \cup V_2$ such that $V_i = \{v_j \in V(G) \mid j \equiv i \pmod{3}\}$. Observe that for each pair of vertices $x, y \in V_i$ there are at least two adjacent vertices from $V(G) \setminus V_i$ between x and y . Hence $G[V_i] \in \text{Forb}_{\prec}(T - \{u, v\})$, $i = 0, 1, 2$, since any copy of $T - \{u, v\}$ in $G[V_i]$ extends to a copy of T in G . Hence $\chi(G) \leq 3 f_{\prec}(T - \{u, v\})$ and since $G \in \text{Forb}_{\prec}(T)$ was arbitrary we have $f_{\prec}(T) \leq 3 f_{\prec}(T - \{u, v\})$. \square

Remark. *We do not use Reduction Lemma 5 for proving our theorems. Instead, it is only used to derive upper bounds on $f_{\prec}(T)$ for some small ordered forests T on three edges in Section 6.*

5 Proofs of Theorems

5.1 Proof of Theorem 1

We will prove that if an ordered graph H contains a cycle, a tangled path or a bonnet then for each positive integer k there is an ordered graph $G \in \text{Forb}_{\prec}(H)$ with $\chi(G) \geq k$.

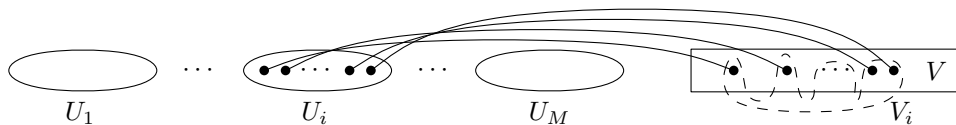


Figure 5: A graph G_k obtained by Tutte's construction from a graph G_{k-1} . Here $G_k[U_i] = G_{k-1}$, $1 \leq i \leq M$.

First assume that H contains a cycle of length ℓ . Fix a positive integer k and consider a graph G of girth at least $\ell + 1$ and chromatic number at least k that exists by [11]. Then no ordering of the vertices of G gives an ordered subgraph isomorphic to H . This shows that for any positive integer k , $f_{\prec}(H) \geq k$ and hence $f_{\prec}(H) = \infty$.

A tangled path is minimal if it does not contain a proper subpath that is tangled. Next we shall show that for each minimal tangled path P and each $k \geq 1$ there is an ordered graph $G_k \in \text{Forb}_{\prec}(P)$ with $\chi(G_k) \geq k$.

By reversing P if necessary we assume that in P the paths Pu and uP cross for the rightmost vertex u in P . We will prove the claim by induction on k . If $k \leq 3$ let $G_k = K_k$ that has no crossing edges and thus no tangled paths. Consider $k \geq 4$ and let G_{k-1} denote an n -vertex graph of chromatic number at least $k - 1$ that does not contain a copy of P . Such a graph exists by induction. The following construction is due to Tutte (alias Blanche Descartes) for unordered graphs [9]. Let $N = (k - 1)(n - 1) + 1$ and $M = \binom{N}{n}$. Consider pairwise disjoint sets of vertices U_1, \dots, U_M, V such that $|U_i| = n$, $i = 1, \dots, M$, $|V| = N$ and $U_1 \prec \dots \prec U_M \prec V$. Let V_1, \dots, V_M be the n -element subsets of V . Let each U_i , $i = 1, \dots, M$, induce a copy of G_{k-1} . Finally let there be a perfect matching between U_i and V_i such that the j^{th} vertex in U_i is matched to the j^{th} vertex in V_i , $i = 1, \dots, M$. See Figure 5.

First we shall show that $\chi(G_k) \geq k$. If there are at most $k - 1$ colors assigned to the vertices of G_k , then by Pigeonhole Principle there are n vertices of V of the same color, i.e., there is a set V_i with all vertices of the same color, say color 1. Since each vertex of U_i is adjacent to a vertex in V_i , no vertex in U_i is colored 1, so if the coloring is proper, then $G[U_i]$ uses at most $k - 2$ colors. Hence the coloring is not proper, since $\chi(G[U_i]) = \chi(G_{k-1}) \geq k - 1$. Therefore $\chi(G_k) \geq k$.

Now, we shall show that G_k does not contain a copy of P . Assume that there is such a copy P' of P in G_k with rightmost vertex u of P' . Let x and y be the neighbors of u in P' , i.e., P' is a union of paths $P'yu$ and uxP' . Then $u \in V$ and $x, y \notin V$, since $G[U_i]$ does not contain a copy of P and there are no edges in $G_k[V]$. Let $x \in U_i$ and $y \in U_j$. Note that $i \neq j$ because the edges between U_i and V form a matching. The path uxP' is a proper subpath of P' and hence is not tangled. Recall that for each edge zw with $z \in U_i$, $w \in V$, and $w \prec u$, we have $z \prec x$ due to the construction of the matching between U_i and V_i . Hence the path uxP' does not contain any vertex $w \in V$ with $w \prec u$, since otherwise the path $uxP'w$ has a vertex left of x contradicting Lemma 4.1 applied to u , x and w . Hence $V(xP') \subseteq U_i$, because there are no edges between U_i 's and u is rightmost in P' . See Figure 6. Similarly, all vertices of $P'y$ are contained in U_j . Thus $P'u$ and uP' do not cross.



Figure 6: A path in G_k with rightmost vertex $u \in V$ is not tangled if Pu and uP are not tangled.

However, P' is a copy of P with respective subpaths crossing, a contradiction. Hence $G_k \in \text{Forb}_{\prec}(P)$.

Now, if an ordered graph H contains a tangled path, then it contains a minimal tangled path. Thus $f_{\prec}(H) = \infty$.

Now, let B be a bonnet. By reversing B if necessary, we assume that B has vertices $u \prec v \preceq x, y \preceq w$ and edges uv, uw, xy . A shift graph $S(n)$ is defined on vertices $\{(i, j) \mid 1 \leq i < j \leq n\}$ and edges $\{(i, j), (j, t) \mid 1 \leq i < j < t \leq n\}$. We will show that some ordering of $S(n)$ does not contain B . Let $G = S(n)$ be a shift graph with vertices ordered lexicographically, i.e., $(x_1, x_2) \prec (y_1, y_2)$ if and only if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. Assume that G contains vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $w = (w_1, w_2)$ that form a copy of B with $u \prec v \preceq x, y \preceq w$ and edges uv, uw, xy . Then $u_2 = v_1$, $u_2 = w_1$, $x_2 = y_1$. Thus $v_1 = w_1$. However, since $v \preceq x, y \preceq w$, we have that $v_1 \leq x_1, y_1 \leq w_1$, so $x_1 = y_1 = v_1 = w_1$. But $x_2 = y_1$, thus $x_2 = x_1$, a contradiction. Thus $G \in \text{Forb}_{\prec}(B)$. We claim that $\chi(G) \geq \log(n) \geq c \log |V(G)|$. Indeed consider a proper coloring ϕ of G using $\chi(G)$ colors and sets of colors $\Phi_i = \{\phi(i, j) \mid i < j \leq n\}$, $1 \leq i \leq n$. Then $\phi(i, j) \notin \Phi_j$, since a vertex (i, j) is adjacent to all vertices (j, t) , $j < t \leq n$. Therefore $\Phi_i \neq \Phi_j$ for all $j < i$. Hence all the sets of colors are distinct. This shows that $2^{\chi(G)} \geq n$, since there are at most $2^{\chi(G)}$ distinct subsets of colors. This proves that $\chi(G) \geq \log(n)$. Thus, for any k , there is an ordered graph of chromatic number at least k in $\text{Forb}_{\prec}(B)$. So, if an ordered graph H contains a bonnet, then $f_{\prec}(H) = \infty$. \square

5.2 Proof of Theorem 2

Let T' be a segment of an ordered tree that does not contain a bonnet or a tangled path. We shall prove that T' is monotonically alternating by induction on $k = |V(T')|$. Every ordered tree on at most two vertices is monotonically alternating. So suppose $k \geq 3$. We have $\chi_{\prec}(T') = 2$ due to Lemma 4.3.

Claim. *The leftmost or the rightmost vertex in T' is of degree 1.*

Proof of Claim. As T' is a segment of an ordered tree, the subgraph induced by $V(T')$ is connected and thus both the leftmost vertex u and the rightmost vertex v in T' are of degree at least 1. For the sake of contradiction assume that both u and v are of degree at least 2. If u and v are adjacent then the edge uv , another edge incident to u and another edge incident v form a tangled path since $\chi_{\prec}(T') = 2$, a contradiction. If u and v are not adjacent let P denote the path in T' connecting u and v . It uses at most one of the edges incident to u . Then any other edge zu

incident to u crosses the edge in P that is incident to v since $\chi_{\prec}(T') = 2$. Hence zP forms a tangled path, a contradiction. This shows that at least one of u or v is a leaf in T' . \triangle

By reversing T' if necessary we assume that the leftmost vertex u is a leaf in T' . The ordered tree $T' - u$ is monotonically alternating by induction and Lemma 4.2. Consider the partition $V(T') = L \dot{\cup} R$, with $L \prec R$ and L and R being independent sets. Such a partition is unique since T' is connected. Let v be the neighbor of u in T' . Since $\chi_{\prec}(T') = 2$, $v \in R$. Since T' is connected, $k \geq 3$ and u is leftmost in T' , the edge uv is not the shortest edge incident to v . Hence $uv \notin S(R)$ and therefore $S(R)$ has no crossing edges by induction. Clearly $uv \in S(L)$ since uv is the only edge incident to u and thus it is the shortest edge incident to u . If uv crosses some edge xy in T' , $x \prec y$, then all vertices in the path connecting v and x are between x and y due to Lemma 4.1 applied to x , y and v . Therefore xy is not the shortest edge incident to x and hence $xy \notin S(L)$. This shows that $S(L)$ has no crossing edges and thus T' is monotonically alternating.

The other way round assume that each segment of an ordered tree T is monotonically alternating. We need to show that each segment contains neither a bonnet nor a tangled path. Let T' denote a segment of T , $V(T') = L \cup R$, $L \prec R$ and $E(T') = S(L) \cup S(R)$, so each edge is either a shortest edge incident to a vertex in R or a shortest edge incident to a vertex in L . Then $\chi_{\prec}(T') \leq 2$ and hence T' does not contain a bonnet. We will prove that T' does not contain a tangled path by induction on $k = |V(T')|$. If $k \leq 3$, then there are no crossing edges in T' and hence no tangled path. Suppose $k \geq 4$.

Assume that the leftmost vertex u and the rightmost vertex w in T' are of degree at least 2. If $uw \in E(T')$ then $uw \notin S(L)$ and $uw \notin S(R)$, a contradiction. So, $uw \notin E(T')$. Consider the longest edge xw incident to w . Then $x \neq u$ and since $xw \notin S(R)$, $xw \in S(L)$. Then the shortest edge incident to u crosses xw , a contradiction since $S(L)$ does not contain crossing edges. Hence the leftmost or the rightmost vertex is a leaf in T' .

By reversing T' if necessary we assume that the leftmost vertex u is a leaf. We see that $T' - u$ is monotonically alternating, thus by induction it does not contain a tangled path. Hence if T' has a tangled path P , then P contains an edge uv crossing some other edge in P , where v is the neighbor of u in T' . Then the rightmost vertex r in P is of degree 2 and to the right of v , since P is tangled and u is leftmost and of degree 1 in T' . Let x and y , $x \prec y$, be neighbors of r in P . Then xr is the shortest edge incident to x , since any shorter edge forms a tangled path with r and y in $T' - u$. This is a contradiction since uv and xr cross and T' is monotonically alternating. Thus T' has no tangled path.

Finally we prove the last statement of the theorem. If H is a connected ordered graph with $f_{\prec}(H) \neq \infty$, then H is a tree that contains neither a bonnet nor a tangled path due to Theorem 1. Hence each segment of H is a monotonically alternating tree. \square

5.3 Proof of Theorem 3

Let T be a non-crossing ordered graph such that $f_{\prec}(T) \neq \infty$. Then T is acyclic, contains no tangled path and no bonnet by Theorem 1. Hence T is a non-crossing ordered forest with no bonnet.

On the other hand let T be a non-crossing forest with no bonnet. Recall that $f_{\prec}(H) \geq k - 1$ for each ordered k -vertex graph H because $K_{k-1} \in \text{Forb}_{\prec}(H)$. We shall prove that $f_{\prec}(T) \neq \infty$. Let $k = |V(T)|$ and consider any ordered graph $G \in \text{Forb}_{\prec}(T)$. We will prove by induction on k that $\chi(G) \leq 2^k$ and $\chi(G) \leq 2k - 3$ if T is a tree. If $k = 2$, then clearly $\chi(G) = 1$. So consider $k \geq 3$.

If T is a tree, then each segment of T is a monotonically alternating tree, by Theorem 2. If there is only one segment in T , then $f_{\prec}(T) \leq 2k - 3$ by Lemma 4.4. If there is more than one segment in T , then there is an inner cut vertex splitting T into two trees T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have $f_{\prec}(T) \leq f_{\prec}(T_1) + f_{\prec}(T_2) \leq 2|V(T_1)| - 3 + 2|V(T_2)| - 3 = 2(|V(T)| + 1) - 6 = 2k - 4$.

If T is a forest we consider several cases. If T has more than one segment, then there is an inner cut vertex splitting T into two forests T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have $f_{\prec}(T) \leq f_{\prec}(T_1) + f_{\prec}(T_2) \leq 2^{|V(T_1)|} + 2^{|V(T_2)|} = 2^t + 2^{k+1-t} \leq 2^k$ with $t = |V(T_1)| \geq 2$. If T has an isolated vertex u , then by Reduction Lemma 2 and induction we have $f_{\prec}(T) \leq 2f_{\prec}(T - u) \leq 2 \cdot 2^{k-1} = 2^k$. Finally, if T has no isolated vertices and exactly one segment, then consider the leftmost and rightmost vertices u and v of T . Since u and v are not isolated in this case, and T is non-crossing with no inner cut vertices, uv is an edge. If uv is isolated, then $k \geq 4$ (since there is no isolated vertex) and by Reduction Lemma 3 and induction we have $f_{\prec}(T) \leq 2 \cdot f_{\prec}(T - \{u, v\}) + 1 \leq 2 \cdot 2^{k-2} + 1 \leq 2^k$. If uv is not isolated, then either u or v , say u , is a leaf of T , since T is non-crossing and does not contain a bonnet. Let xv denote the longest edge incident to v in $T - u$. Note that x exists since the edge uv is not isolated. Then there is no other vertex between u and x , since such a vertex would be isolated in the non-crossing forest T without bonnets. Thus, u is a reducible vertex, so by Reduction Lemma 4 and induction we have $f_{\prec}(T) \leq 2f_{\prec}(T - u) \leq 2 \cdot 2^{k-1} = 2^k$.

Next, we provide a k -vertex non-crossing tree with no bonnet such that $\infty \neq f_{\prec}(T) \geq k$. Let T be a monotonically alternating path on $k \geq 4$ vertices with leftmost vertex of degree 1, as in Figure 7 (right). Further let G denote a graph on vertices $u \prec x_1 \prec \dots \prec x_{k-2} \prec y_1 \prec \dots \prec y_{k-2} \prec x \prec y$ such that xy is an edge and $\{u, x_1, \dots, x_{k-2}\}$, $\{u, y_1, \dots, y_{k-2}\}$, $\{x, x_1, \dots, x_{k-2}\}$, and $\{y, y_1, \dots, y_{k-2}\}$ induce complete graphs on $k - 1$ vertices each. See Figure 7 (left).

We shall show that $G \in \text{Forb}_{\prec}(T)$ and $\chi(G) \geq k$. Consider a proper vertex

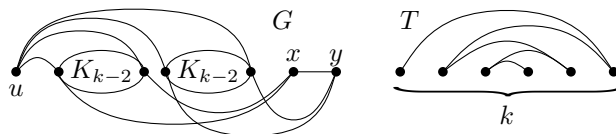


Figure 7: An ordered graph G with chromatic number k not containing a non-crossing and ordered tree T on k vertices without bonnets on the right, $k = 6$.

coloring of G using colors $1, \dots, k-1$. Without loss of generality u has color 1. Then all colors $2, \dots, k-1$ are used on the vertices x_1, \dots, x_{k-2} as well as on y_1, \dots, y_{k-2} . Hence both x and y are of color 1, a contradiction. Thus $\chi(G) \geq k$.

Assume that there is a copy P of T in G . Let v be the leftmost and w be the rightmost vertex in P . Note that vw is an edge and that there are k vertices between v and w (including v and w). Therefore vw is one of the edges uy_i , $1 \leq i \leq k-2$, x_jx , $1 \leq j \leq k-2$, or y_1y . In the first case $V(P) \subseteq \{u, y_1, \dots, y_{k-2}\}$, in the second case $V(P) \subseteq \{x_1, \dots, x_{k-2}, x\}$ and in the last case either $P = y_1, y, x$ or $V(P) \subseteq \{y, y_1, \dots, y_{k-2}\}$. Since T has at least 4 vertices, $P \neq y_1, y, x$. So in any case P has at most $k-1$ vertices, a contradiction since T has k vertices. Hence $G \in \text{Forb}_{\prec}(T)$.

Finally it is easy to see that $f_{\prec}(T) = k-1$ for any ordered tree T on at most 3 vertices using Reduction Lemmas 1 and 4. \square

5.4 Proof of Theorem 4

- Let T be an ordered forest on k vertices where each segment is a generalized star, a 2-nesting, or a 2-crossing. Let T_1, \dots, T_s denote the segments of T and $k_i = |V(T_i)|$, $1 \leq i \leq s$. Let T' be a segment of T . If T' is a generalized star on k' vertices, then the center of the star is leftmost (or rightmost) in T' . Let $G \in \text{Forb}_{\prec}(T')$. Then each vertex in G has at most $k'-2$ neighbors to the right (or to the left). Thus each such graph can be greedily colored from right to left (or left to right) with at most $k'-1$ colors. This shows that $f_{\prec}(T') \leq |V(T')| - 1$. If T' is a 2-nesting, then $f_{\prec}(T') = 3 = |V(T')| - 1$ due to [10] (Lemma 9). If T' is a 2-crossing, then $f_{\prec}(T') = 3 = |V(T')| - 1$, since any graph not containing T' is outerplanar and outerplanar graphs have chromatic number at most 3. We apply Reduction Lemma 1 and the results above which yield $f_{\prec}(T) \leq \sum_{i=1}^s f_{\prec}(T_i) \leq \sum_{i=1}^s (k_i - 1) = k - 1$.
- Let T be an ordered forest on k vertices where each segment is a generalized star, a non-crossing tree without bonnets, a crossing or a nesting. Let T_1, \dots, T_s denote the segments of T and $k_i = |V(T_i)| \geq 2$. Let T' be a segment of T . If T' is a k' -nesting or a k' -crossing, $k' \geq 2$, then $f_{\prec}(T') \leq 4(k'-1) \leq 2|V(T')| - 3$ due to equation (1), since any graph $G \in \text{Forb}_{\prec}(T')$ contains less than $2(k'-1)|V(G)|$ edges due to Dujmovic and Wood [10] (for nestings), respectively Capoleas and Pach [5] (for crossings).

Further $f_{\prec}(T') \leq 2|V(T')| - 3$ if T' is a non-crossing tree without bonnets due to Theorem 3. Hence Reduction Lemma 1 yields $f_{\prec}(T) \leq \sum_{i=1}^s f_{\prec}(T_i) \leq \sum_{i=1}^s (2k_i - 3) \leq 2k - 3$.

- Let $T = M(t, m, \pi)$ for some positive integers m and t and a permutation π of $[t]$. If $t = 1$, then $f_{\prec}(T) = m$ due to the results above, since $M(1, m, \pi)$ is a star on $m + 1$ vertices. Weidert [19] proves that $\text{ex}_{\prec}(n, M(t, 1, \pi)) \leq \text{ex}_{\prec}(n, M(t, 2, \pi)) \leq 11t^4 \binom{2t^2}{2t} n < t^4 (2t^2)^{2t} n$ for any positive integer $t \geq 2$ and any permutation π of $[t]$. Moreover if $m \geq 2$, then

$$\text{ex}_{\prec}(n, M(t, m, \pi)) \leq 2^{t(m-2)} \text{ex}_{\prec}(n, M(t, 2, \pi))$$

due to a reduction by Tardos [18]. Therefore $\text{ex}_{\prec}(n, M(t, m, \pi)) < 2^{tm} t^{4+4t} n$. Thus, using the fact that $|V(T)| = k = tm + t$ and equation (1) we have that $f_{\prec}(M(t, m, \pi)) \leq 2^{tm+9t \log(t)} \leq 2^{10k \log k}$.

- Conlon *et al.* [7] and independently Balko *et al.* [2] prove that there is a positive constant c such that for any sufficiently large positive integer k there is an ordered matching on k vertices with ordered Ramsey number at least $2^{c \frac{\log(k)^2}{\log \log(k)}}$. If, for some ordered graph H , the edges of a complete ordered graph G on $N = R_{\prec}(H) - 1$ vertices are colored in two colors without monochromatic copies of H , then both color classes form ordered graphs G_1 and G_2 in $\text{Forb}_{\prec}(H)$. Then one of the G_i 's has chromatic number at least \sqrt{N} , since a product of proper colorings of G_1 and G_2 yields a proper coloring of G using $\chi(G_1)\chi(G_2) \geq \chi(G) = N$ colors. This shows that there is a positive constant c' such that for all positive integers k and ordered matchings H on k vertices with $f_{\prec}(H) \geq 2^{c' \frac{\log(k)^2}{\log \log(k)}}$. \square

6 Small Forests

Let P_k denote a path on k vertices, M_k a matching on k edges and S_k a star with k leaves (note that $M_1 = S_1 = P_2$ and $P_3 = S_2$). Further let $G + H$ denote the vertex disjoint union of graphs G and H . Then the set of all forests without isolated vertices and at most 3 edges is given by

$$\{P_2, S_2, M_2, S_3, P_4, S_2 + P_2, M_3\}.$$

Let G denote a graph on n vertices and a automorphisms. Then the number $\text{ord}(G)$ of non-isomorphic orderings of G equals $\text{ord}(G) = \frac{n!}{a}$. Hence

$$\begin{aligned} \text{ord}(P_2) &= \frac{2!}{2} = 1, & \text{ord}(S_2) &= \frac{3!}{2} = 3, & \text{ord}(M_2) &= \frac{4!}{8} = 3, & \text{ord}(S_3) &= \frac{4!}{3!} = 4, \\ \text{ord}(P_4) &= \frac{4!}{2} = 12, & \text{ord}(S_2 + P_2) &= \frac{5!}{2 \cdot 2} = 30, & \text{ord}(M_3) &= \frac{6!}{6 \cdot 4 \cdot 2} = 15. \end{aligned}$$

Recall that the reverse \bar{T} of an ordered graph T is the ordered graph obtained by reversing the ordering of the vertices in T . Note that $f_{\prec}(T) = f_{\prec}(\bar{T})$ for any

ordered graph T since $G \in \text{Forb}_{\prec}(T)$ if and only if $\overline{G} \in \text{Forb}_{\prec}(\overline{T})$. Table 8 shows all ordered forests T without isolated vertices and at most 3 edges and their f_{\prec} values, where only one of T and \overline{T} is listed. So when T and \overline{T} are not isomorphic ordered graphs the entry in the table represents two graphs. Such cases are marked with an $*$. For example there are only two instead of three entries for S_2 and similarly for the other graphs.

7 Conclusions

In this paper, we consider the function $f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\}$ for ordered graphs H on at least 2 vertices. We prove that in contrast to unordered and directed graphs, $f_{\prec}(H) = \infty$ for some ordered forests H . To this end we explicitly describe several infinite classes of minimal ordered forests H with $f_{\prec}(H) = \infty$. A full answer to the following question remains open.

Question 1. *For which ordered forests H does $f_{\prec}(H) = \infty$ hold?*

We completely answer Question 1 for non-crossing ordered graphs H . Suppose that H is a non-crossing ordered k -vertex graph with $f_{\prec}(H) \neq \infty$. We prove that, if H connected, then $k - 1 \leq f_{\prec}(H) \leq 2k - 3$ and, if H is disconnected, then $k - 1 \leq f_{\prec}(H) \leq 2^k$. In addition, we give infinite classes of graphs for which $f_{\prec}(H) = |V(H)| - 1$, as well as infinite classes of graphs for which $|V(H)| \leq f_{\prec}(H) \neq \infty$. Note that we do not know whether $f_{\prec}(H) \neq \infty$ for the matchings in the last statement of Theorem 4. For crossing connected ordered graphs, we reduce Question 1 to monotonically alternating trees:

Question 2. *For which monotonically alternating trees H does $f_{\prec}(H) = \infty$ hold?*

We do not have an answer to Question 2 even for some monotonically alternating paths. A smallest unknown such path is $u_5u_1u_3u_2u_4$, where $u_1 \prec \dots \prec u_5$. See Figure 9 (left). The situation becomes even more unclear for crossing disconnected graphs. We do not know the value of $f_{\prec}(H)$ for some ordered matchings H . A smallest such matching has edges u_1u_3 , u_2u_5 and u_4u_6 where $u_1 \prec \dots \prec u_6$. See Figure 9 (right). Note that Reduction Lemmas 1, 2, 3 and 4 apply to crossing ordered graph as well. We find a more precise version of Reduction Lemma 2 and other types of reductions, similar to reductions for matrices in [18], but none of these lead to significantly better upper bounds in Theorems 3 and 4 or a new class of forests with finite f_{\prec} . The following question remains open, even when restricted to non-crossing graphs.

Question 3. *For $k \geq 4$, what is the value of the function*

$$f_{\prec}(k) = \max\{f_{\prec}(H) \mid |V(H)| = k, f_{\prec}(H) \neq \infty\}?$$

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\mathbf{T}						
$f_{\prec}(\mathbf{T})$	1 (Thm. 4)	2 * (Thm. 4)	2 (Thm. 4)	3 (Thm. 4)	3 (Thm. 4)	3 (Thm. 4)
\mathbf{T}						
$f_{\prec}(\mathbf{T})$	3 * (Thm. 4)	3 * (Thm. 4)	3 (Thm. 4)	∞ * (bonnet)	∞ * (tangled)	
\mathbf{T}						
$f_{\prec}(\mathbf{T})$	3 * (Thm. 4)	∞ (bonnet)	∞ (tangled)	4 * (Lem. 4.4, Fig. 7)	≤ 4 (Red. 4)	
\mathbf{T}						
$f_{\prec}(\mathbf{T})$	4 * (Thm. 4)	≤ 6 * (Red. 4)	? * (Red. 3)	≤ 6 * (Red. 3)	≤ 6 * (Lem. 4.4)	
\mathbf{T}						
$f_{\prec}(\mathbf{T})$? * (Thm. 4)	$\neq \infty$ * (Thm. 4)	∞ * (bonnet)	? * (Thm. 4)	4 * (Thm. 4)	
\mathbf{T}						
$f_{\prec}(\mathbf{T})$	4 * (Thm. 4)	4 * (Thm. 4)	? * (Thm. 4)	≤ 6 (Red. 3)	4 * (Thm. 4)	? (Thm. 4)
\mathbf{T}						
$f_{\prec}(\mathbf{T})$	5 (Thm. 4)	5 * (Thm. 4)	5 * (Thm. 4)	≤ 9 * (Red. 5)	≤ 7 (Red. 3)	
\mathbf{T}						
$f_{\prec}(\mathbf{T})$? (Thm. 4)	$\neq \infty$ * (Thm. 4)	≤ 9 (Red. 5)	≤ 8 (Thm. 4)	≤ 7 (Red. 3)	≤ 8 (Thm. 4)

Figure 8: All ordered forests T on at most 3 edges without isolated vertices and their f_{\prec} value.

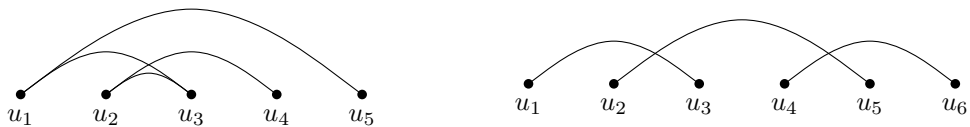


Figure 9: Ordered graphs H for which we don't know whether $f_{\prec}(H) = \infty$.

References

- [1] L. Addario-Berry, F. Havet, C. L. Sales, B. Reed, and S. Thomassé. Oriented trees in digraphs. *Discrete Math.*, 313(8):967–974, 2013.
- [2] M. Balko, J. Cibulka, K. Král, and J. Kynčl. Ramsey numbers of ordered graphs. *Electronic Notes in Discrete Mathematics*, 49:419 – 424, 2015. The Eight European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2015.
- [3] P. Brass, G. Károlyi, and P. Valtr. A Turán-type extremal theory of convex geometric graphs. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, pages 275–300. Springer, Berlin, 2003.
- [4] S. A. Burr. Subtrees of directed graphs and hypergraphs. In *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. I*, volume 28 of *Congr. Numer.*, pages 227–239, 1980.
- [5] V. Capoleas and J. Pach. A Turán-type theorem on chords of a convex polygon. *J. Combin. Theory Ser. B*, 56(1):9–15, 1992.
- [6] V. Chvátal. Perfectly ordered graphs. In *Topics on perfect graphs*, volume 88 of *North-Holland Math. Stud.*, pages 63–65. North-Holland, Amsterdam, 1984.
- [7] D. Conlon, J. Fox, C. Lee, and B. Sudakov. Ordered Ramsey numbers. *J. Combin. Theory Ser. B*. to appear.
- [8] P. Damaschke. Forbidden ordered subgraphs. In *Topics in combinatorics and graph theory (Oberwolfach, 1990)*, pages 219–229. Physica, Heidelberg, 1990.
- [9] B. Descartes. A three colour problem. *Eureka*, 21, 1947.
- [10] V. Dujmovic and D. R. Wood. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.
- [11] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.
- [12] Z. Füredi and P. Hajnal. Davenport-Schinzel theory of matrices. *Discrete Math.*, 103(3):233–251, 1992.
- [13] M. Ginn. Forbidden ordered subgraph vs. forbidden subgraph characterizations of graph classes. *J. Graph Theory*, 30(2):71–76, 1999.

- [14] M. Klazar. Extremal problems for ordered (hyper) graphs: applications of Davenport-Schinzel sequences. *European J. Combin.*, 25(1):125–140, 2004.
- [15] M. Klazar. Extremal problems for ordered hypergraphs: small patterns and some enumeration. *Discrete Appl. Math.*, 143(1-3):144–154, 2004.
- [16] A. Marcus and G. Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.
- [17] J. Pach and G. Tardos. Forbidden paths and cycles in ordered graphs and matrices. *Israel J. Math.*, 155:359–380, 2006.
- [18] G. Tardos. On 0-1 matrices and small excluded submatrices. *J. Combin. Theory Ser. A*, 111(2):266–288, 2005.
- [19] C. Weidert. Extremal problems in ordered graphs. Master’s thesis, Simon Fraser University, 2009. arXiv:0907.2479.
- [20] D. B. West. *Introduction to graph theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996.