

# Graphs admitting $d$ -realizers: spanning-tree-decompositions and box-representations

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## Abstract

A  $d$ -realizer is a collection  $R = \{\pi_1, \dots, \pi_d\}$  of  $d$  permutations of a set  $V$  representing an antichain in  $\mathbb{R}^d$ . We use  $R$  to define a graph  $G_R$  on the suspended set  $V^+ = V \cup \{s_1, \dots, s_d\}$ . It turns out that  $G_R$  has  $dn + \binom{d}{2}$  edges ( $n = |V|$ ), among them the edges of the outer clique on  $\{s_1, \dots, s_d\}$ . The inner edges of  $G_R$  can be partitioned into  $d$  trees such that  $T_i$  spans  $V + s_i$ . In the case  $d = 3$  the graph  $G_R$  is a planar triangulation and  $T_1, T_2, T_3$  is a Schnyder wood on  $G_R$ . The following two results show that  $d$ -realizers resemble Schnyder woods in several aspects:

- Complete point-face contact systems of homothetic simplices in  $\mathbb{R}^{d-1}$  induce a  $d$ -realizer.
- Any spanning subgraph of a graph  $G$  with a  $d$ -realizer has a  $d$ -dimensional proper touching box representation.

We expect that  $d$ -realizers will prove to be valuable generalization of Schnyder woods to higher dimensions.

## 1 Introduction

We consider  $\mathbb{R}^d$  equipped with the *dominance order*, i.e., for  $x, y \in \mathbb{R}^d$  we have  $x \leq_{\text{dom}} y$  if and only if  $x_i \leq y_i$  for  $i = 1, \dots, d$ . A set  $P \subset \mathbb{R}^d$  is in *general position* if no two points of  $P$  share a coordinate. If no two points of a set  $P$  are in the dominance relation  $\leq_{\text{dom}}$ , then we call  $P$  an *antichain*. If  $P$  is in general position, then the projection to the  $i$ th coordinate yields a permutation  $\pi_i$  of  $P$ . In compliance with the previous definition, we call a family of permutations  $\pi_1, \dots, \pi_d$  of  $V$  an *antichain* if for all  $x, y \in V$  there are indices  $i$  and  $j$  such that  $x$  precedes  $y$  in  $\pi_i$  and  $y$  precedes  $x$  in  $\pi_j$ . We use the notation  $x \prec_i y$  to denote that  $x$  precedes  $y$  in  $\pi_i$ .

An antichain  $V$  in  $\mathbb{R}^d$  is *suspended* if  $V$  contains a suspension vertex for each  $i$ , i.e., a vertex  $s_i = (0, \dots, 0, M_i, 0, \dots, 0)$  and  $0 \leq v_i < M_i$  for all  $v \in V \setminus s_i$ .

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Similarly  $s_i$  is an  $i$ -suspension for  $\pi_1, \dots, \pi_d$  if  $s_i$  is the last element of  $\pi_i$  and among the first  $d - 1$  elements in  $\pi_j$  for  $j \neq i$ . The family  $\pi_1, \dots, \pi_d$  is suspended if it has an  $i$ -suspension for each  $i \in [d]$ .

**Definition 1** A  $d$ -realizer is a suspended antichain  $\pi_1, \dots, \pi_d$  of permutations of  $V^+$  where  $V^+ = V \cup S$  and  $S = \{s_1, \dots, s_d\}$  is the set of suspensions.

**Definition 2** The graph of a  $d$ -realizer  $(\pi_1, \dots, \pi_d)$  is the graph  $G_R = (V^+, E^+)$  with  $E^+ = E_R \cup E_S$  where  $E_S$  is the set of edges of a clique on  $S$  and pairs  $x, y$  are edges in  $E_R$  if they satisfy two properties:

$(x, y)$  is a **candidate pair**: for all  $z \neq x, y$  there is an  $i$  with  $x \prec_i z$  and  $y \prec_i z$ .

$(x, y)$  has the **1-of- $d$ -property**: there is a unique  $i \in [d]$  with  $x \prec_i y$ , i.e.,  $y \prec_j x$  for all  $j \neq i$ .

The definition of Schnyder woods was originally motivated by the study of the order dimension of incidence posets of graphs. In this line of research the following definition was proposed in [8]:

The dimension of  $G = (V, E)$  is at most  $k$  if there are permutations  $\pi_1, \dots, \pi_k$  of  $V$  such that each edge  $(x, y) \in E$  is a candidate pair.

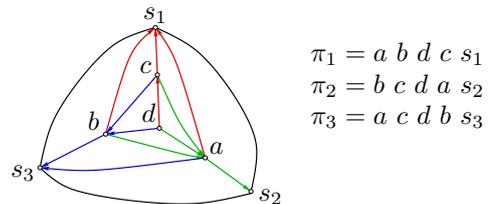
If  $G$  is two-connected, then it follows that  $\pi_1, \dots, \pi_k$  is an antichain. The following are known:

- $\dim(G) \leq 3$  iff  $G$  is planar (Schnyder [12]).
- $\dim(G) \leq 4 \implies G$  has at most  $3/8n^2$  edges.
- Exact values of  $\dim(K_n)$  are known for  $n < 10^{40}$ .

The 1-of- $d$ -property naturally leads to a coloring and an orientation of the edges of  $G_R$ : The orientation is  $x \rightarrow y$  if  $x$  precedes  $y$  only in a single  $\pi_i$ . The color of  $x \rightarrow y$  is the index  $i$  with  $x \prec_i y$ . Let  $T_i$  be the set of edges of color  $i$ .

Note that in the case  $d = 3$  the 1-of-3-property is fulfilled by all candidate edges; this is where Schnyder's coloring and orientation of edges comes from. Schnyder [12] found that for all  $i$  the following two properties hold:

- (a)  $T_i$  is an in-arborescence with root  $s_i$ .
- (b)  $T_{i-1} + T_{i+1} + T_i^{-1}$  is acyclic.



**Fig. 1:** An example of a 3-realizer and its graph.

In the next section we show that this also holds in the case of a  $d$ -realizer. In Section 3 we continue to show how  $d$ -realizer can be used to construct proper touching box representations; the  $d = 3$  case of this result was obtained in [1]. In Section 4 we connect  $d$ -realizers to orthogonal surfaces and show how they arise from touching simplices. We conclude with examples and some open problems.

## 2 Spanning-tree-decompositions

**Proposition 1** *Let a graph be defined by a  $d$ -realizer  $(\pi_1, \dots, \pi_d)$ . If  $T_i$  is the set of edges of color  $i$ , then  $T_i$  is an in-arborescence with root  $s_i$ .*

*Proof.* We first show that each  $v \in V$  has a unique out-edge in  $T_i$ .

Let  $H_i(x)$  be the set of all  $y$  with  $x \prec_i y$  and  $y \prec_j x$  for all  $j \neq i$ , i.e., the set of all  $y$  such that the pair  $(x, y)$  has the 1-of- $d$ -property. Since the pair  $(x, s_i)$  has the 1-of- $d$ -property  $H_i(x) \neq \emptyset$  for all  $v \in V$ . Let  $p_i(x)$  be the first element of  $H_i(x)$  with respect to  $\pi_i$ , i.e.,  $p_i(x)$  is the least element of  $\pi_i$  such that  $(x, p_i(x))$  has the 1-of- $d$ -property.

**Claim 1.**  $(x, p_i(x))$  is a candidate.

Consider  $z \neq x, p_i(x)$ . Since a  $d$ -realizer is an antichain there is some  $j$  with  $x \prec_j z$ . If  $j \neq i$ , then  $p_i(x) \prec_j z$  and by transitivity  $p_i(x) \prec_j z$ . If the only choice for  $j$  is  $i$ , then  $z \in H_i(x)$  and  $p_i(x) \prec_j z$  follows from the choice of  $p_i(x)$ .  $\triangle$

From Claim 1 it follows that  $(x, p_i(x)) \in T_i$ .

**Claim 2.** If  $(x, y)$  is a candidate with  $y \in H_i(x)$ , then  $y = p_i(x)$ .

Indeed if  $y \neq p_i(x)$  then there is no  $\pi_j$  where  $x$  and  $y$  precede  $p_i(x)$ . In  $\pi_i$  we have  $x \prec_i p_i(x) \prec_i y$  and if  $j \neq i$ , then  $p_i(x) \prec_j x$ .  $\triangle$

Hence  $(x, p_i(x))$  is the only out-edge of  $x$  in  $T_i$ . Therefore the number of edges of  $T_i$  is  $|V|$ . Since  $T_i$  is spanning  $V + s_i$  it only remains to show that  $T_i$  is connected. For  $x \in V$  define  $x_0 = x$  and for  $k \geq 0$  let  $x_{k+1} = p_i(x_k)$ . This defines a path that moves to the right on  $\pi_i$ ; hence it must reach  $s_i$ .  $\square$

**Corollary 1** *A graph  $G_R$  defined by a  $d$ -realizer on a vertex set  $V^+$  with  $|V^+| = n + d$  has  $dn + \binom{d}{2}$  edges.*

**Proposition 2** *If  $G_R$  is defined by a  $d$ -realizer, then  $T_i^{-1} + \sum_{j \neq i} T_j$  is acyclic.*

*Proof.* From the 1-of- $d$ -property it follows that directed edges from  $T_j$  with  $j \neq i$  point to the left in the order of vertices given by  $\pi_i$ . The same is true if we revert the direction of the edges of  $T_i$ , i.e., for the directed edges of  $T_i^{-1}$ .  $\square$

## 3 Box-representations

We consider axis aligned boxes in  $d$ -space. Such a *box* is a set  $B(a, b) = \{x \in \mathbb{R}^d : a \leq_{\text{dom}} x \leq_{\text{dom}} b\}$  or equivalently  $B(a, b) = \prod_{i=1}^d [a_i, b_i]$ . The interior of  $B(a, b)$  is  $\{x \in \mathbb{R}^d : a <_{\text{dom}} x <_{\text{dom}} b\} = \prod_{i=1}^d (a_i, b_i)$ . Two boxes  $B$  and  $B'$

are *properly touching* iff they have a unique separating hyperplane  $H = \{x \in \mathbb{R}^d : n_H^T \cdot x = b_H\}$ , i.e.,  $n_H^T \cdot x \leq b_H$  for all  $x \in B$  and  $n_H^T \cdot x \geq b_H$  for all  $x \in B'$ . In other words,  $B$  and  $B'$  are properly touching if their interiors are disjoint and their intersection is  $(d - 1)$ -dimensional.

**Definition 3** *A proper touching box representation of a graph  $G = (V, E)$  in  $d$  dimensions consists of a map  $v \rightarrow B_v$  from the vertices to  $d$ -dimensional boxes with pairwise disjoint interiors, such that boxes  $B_u$  and  $B_v$  are properly touching iff  $(u, v) \in E$ .*

Box representations of graphs have been studied in 2D with different names, e.g as *rectangle contact graphs*. Surveys of the state of the art can be found in [3] and [5].

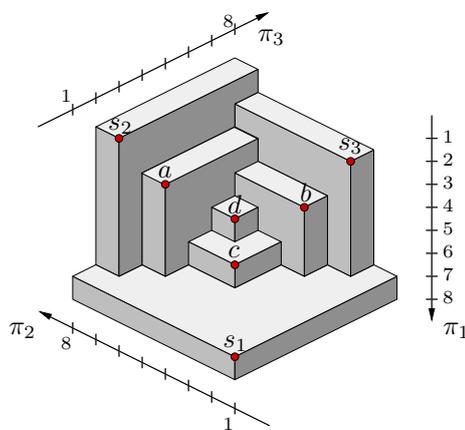
For 3D, Thomassen [13] shows that any planar graph has a proper touching box representation. Felsner and Francis [6] prove that any planar graph has a touching cube representation, if the graph is a subgraph of a 4-connected triangulation the representation is proper. New proofs of Thomassen's result and additional results on cube representations can be found in [1].

**Theorem 1** *Any spanning subgraph  $H$  of a graph  $G$  with a  $d$ -realizer has a  $d$ -dimensional proper touching box representation.*

*Proof.* Let  $(\pi_1, \dots, \pi_d)$  be the  $d$ -realizer for  $G$ . We assume that the order of the first  $d - 1$  elements in  $\pi_i$  (these are suspensions) is  $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_d)$ . This has the advantage that for  $i < j$  the pair  $(s_j, s_i)$  has the 1-of- $d$ -property. The pair is a candidate so we can treat it as a regular edge in  $T_i$ .

With  $\text{rank}_i(x)$  we denote the position of  $x$  in  $\pi_i$ , i.e., if we think of  $\pi_i$  as a bijective map  $\pi_i : [n + d] \rightarrow V^+$ , then  $\text{rank}_i(x) = \pi_i^{-1}(x)$ . For each  $x$  and  $i$  we define  $p_i(x)$  as in Prop 1. For a suspension  $s_i$  and all  $j$  with  $i \leq j$  we assume the value  $n + d + 1$  for the strictly speaking undefined expression  $\text{rank}_j(p_j(s_i))$ .

We now show how to represent  $G$ . The box for vertex  $x$  in  $G$  is  $B(x) = \prod_{i=1}^d [\text{rank}_i(x), \text{rank}_i(p_i(x))]$ .



**Fig. 2:** The proper touching box representation of the graph from Fig. 1 obtained with our method. The view is from below, i.e., the labeled corners are the minima of the boxes.

We need to show proper contact between the box  $B(x)$  and the box  $B(p_i(x))$  for all  $i$ . Let  $y = p_i(x)$ . Since the projection to  $B(x)$  and  $B(y)$  to dimension  $i$  share the point  $\text{rank}_i(y)$ , it suffices to show that  $\text{rank}_j(x) \in (\text{rank}_j(y), \text{rank}_j(p_j(y)))$  for all  $j \neq i$ . By the 1-of- $d$ -property,  $\text{rank}_j(y) < \text{rank}_j(x)$  for all  $j \neq i$ . So it suffices to check that  $\text{rank}_j(x) < \text{rank}_j(p_j(y))$  for all  $j \neq i$ .

Let  $z = p_j(y)$  and suppose  $z \prec_j x$ . By the 1-of- $d$ -property,  $z \prec_k y$  for all  $k \neq j$ . Since  $y \prec_k x$  for all  $k \neq i$  transitivity implies that  $z \prec_k x$  for all  $k \neq i, j$  and by supposition also for  $k = j$ . Since a  $d$ -realizer is an antichain we can conclude that  $x \prec_i z$ .

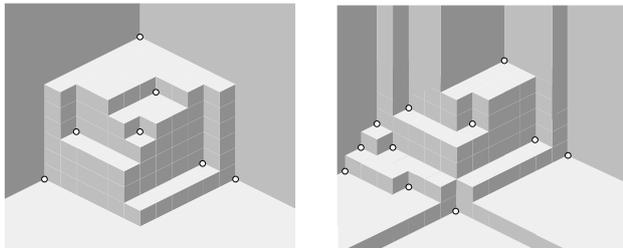
It now happens that  $(x, z)$  and  $(x, y)$  both have the 1-of- $d$ -property and  $x \prec_i z \prec_i y$ . This however contradicts the choice of  $y = p_i(x)$  as the least element of  $\pi_i$  such that  $(x, y)$  has the 1-of- $d$ -property. Therefore  $x \prec_j z$  as needed for the box contact.

To represent a subgraph of  $G$ , remove unneeded boxes and edges from the box representation. To get rid of an edge  $(x, p_i(x))$  change the extent of  $B(x)$  in dimension  $i$  to  $[\text{rank}_i(x), \text{rank}_i(p_i(x)) - \varepsilon]$ .  $\square$

#### 4 Orthogonal surfaces and simplices

In this section we take a more geometric look at the graphs of  $d$ -realizers.

With a point  $p \in \mathbb{R}^d$  we associate its *cone*  $C(p) = \{q \in \mathbb{R}^d : p \leq_{\text{dom}} q\}$ . The *filter*  $\langle V \rangle$  generated by  $V$  is the union of all cones  $C(v)$  for  $v \in V$ . The *orthogonal surface*  $S_V$  generated by  $V$  is the boundary of  $\langle V \rangle$ . A point  $p \in \mathbb{R}^d$  belongs to  $S_V$  if and only if  $p$  shares a coordinate with all  $v \leq_{\text{dom}} p$ ,  $v \in V$ . The generating set  $V$  is an antichain if and only if all elements of  $V$  appear as minima on  $S_V$ .



**Fig. 3:** Two orthogonal surfaces in  $\mathbb{R}^3$ : the left one is generated by a suspended antichain in general position; the antichain generating the right one is neither suspended nor in general position. As usual for orthogonal surfaces we take a view from above, the generating points are the minima of the surface.

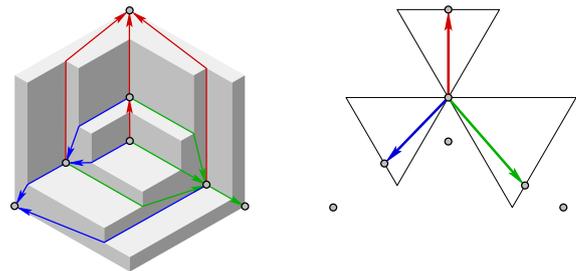
Miller [10] observed the connection between Schnyder woods and orthogonal surfaces in  $\mathbb{R}^3$ . He and subsequently others [4, 9] used orthogonal surfaces to give new proofs for the Brightwell-Trotter theorem about the order dimension of face lattices of 3-polytopes [2]. In fact the dominance order of critical points (maxima, minima, and saddle points) of a 3-dimensional orthogonal surface that is generated by a suspended antichain is the truncated face lattice of a 3-polytope with one facet removed. The converse also holds: every 3-polytope with a facet, selected for removal, has a corresponding orthogonal surface.

The Brightwell-Trotter theorem is an important generalization of Schnyder's dimension theorem. Since orthog-

onal surfaces can be considered in arbitrary dimensions they provide a direction for generalizing Schnyder structures to higher dimensions. This approach has been taken in [7]. The strongest result in the area is a theorem of Scarf [11] that can be restated as follows: the dominance order of critical points of a  $d$ -dimensional orthogonal surface that is generated by a suspended antichain in general position is the truncated face lattice of a simplicial  $d$ -polytope with one facet removed. However, the general situation is not nearly as nice as in 3 dimensions. There are simplicial  $d$ -polytopes that do not have a corresponding orthogonal surface and if we allow non-general position the dominance order of critical points need not even be a truncated lattice [7].

The orthogonal surface view for graphs given by a  $d$ -realizer  $R$  is as follows: Embed vertex  $v$  at the point  $p_v$  whose coordinates are the ranks of  $v$  in the realizer. The out-neighbor of  $v$  in color  $i$  is the vertex  $w$  whose cone  $C(p_w)$  is first hit by the ray leaving  $p_v$  in the  $i$ th coordinate direction.

In the 3-dimensional case we can embed every triangulation (graph with a 3-realizer) on an orthogonal surface  $S_V$  with a coplanar  $V$ , i.e., all  $p \in V$  lie in a plane  $h$  with normal  $\mathbf{1} = (1, 1, 1)$ . Identifying  $h$  with  $\mathbb{R}^2$  we can find the three edges of a vertex  $v$  by growing homothetic equilateral triangles with a corner in  $v$  until they hit another vertex; Fig. 4 shows an example.



**Fig. 4:** The graph from Fig. 1 on a coplanar orthogonal surface and a sketch illustrating how to recover the out-edges of a vertex from the generating set of points in the plane.

In the same way we may use a set of points in  $d$ -space and the homothets of a  $d$ -simplex to build a graph from the class defined by  $(d + 1)$ -realizers. The details are as follows: Let  $\Delta$  be a fixed  $d$ -simplex in  $\mathbb{R}^d$  and let  $P$  be a set of points such that no hyperplane parallel to a facet of  $\Delta$  contains more than one point (this is the appropriate general position assumption). Let  $S$  be the set of corners of a homothet of the dual of  $\Delta$  that contains  $P$ , this is the set of suspensions. Now, for each point  $p \in P$  and each corner  $x$  of  $\Delta$  find the unique point  $q$  such that there is a homothety that maps  $\Delta$  to  $\Delta'$  such that (1) the corner  $x$  of  $\Delta'$  is at  $p$  (2)  $\Delta'$  has no point of  $P$  in the interior and (3)  $q$  is on the boundary of  $\Delta'$ . This condition characterizes the edges  $p \rightarrow q$  of color  $x$  in the graph  $G_\Delta(P)$ .

**Problem 1** Let  $G$  be the graph of a  $d$ -realizer. Is it always possible to find a point set  $P$  in  $\mathbb{R}^{d-1}$  such that  $G = G_\Delta(P)$ ?

There is one class of graphs where we know that the answer to the problem is yes. These are the skeleton graphs

of  $d$ -dimensional stacked polytopes, also known as simple  $d$ -trees. A  $d$ -tree is a graph admitting a stacking sequence, i.e., a listing  $v_1, v_2, \dots, v_n$  of the vertices such that  $v_1, \dots, v_{d+1}$  is a clique and for each  $j > d+1$  the neighbors of  $v_j$  with indices  $< j$  induce a clique  $C_j$  of size  $d$ . A  $d$  tree is simple if  $C_i \neq C_j$  whenever  $i \neq j$ , i.e., each  $d$ -clique can be used at most once for stacking. If  $G$  is a simple  $d$ -tree a corresponding point set  $P$  can be constructed along the stacking sequence.

In fact, besides this class we know only a few examples of graphs that have a  $d$ -realizer with  $d > 3$ . We know that unlike in the  $d = 3$  case we also have non-simple  $d$ -trees in the class: Consider a simple  $d$ -tree with realizer  $(\pi_1, \dots, \pi_d)$  and let  $x$  be a vertex with  $\deg(x) = d$ , for example the last vertex of the construction sequence has this property. Add a new vertex  $x'$  by placing it immediately before  $x$  in  $\pi_1$  and  $\pi_2$  and immediately after  $x$  in all the other  $\pi_j$ . It is easily seen that  $x$  and  $x'$  have the same neighbors in the same colors, in particular they are stacked over the same clique.

**Problem 2** Characterize the  $d$ -trees that have a  $d$ -realizer.

**Problem 3** Find meaningful examples and families of graphs that have a  $d$ -realizer.

Regarding the recognition of graphs that have a  $d$ -realizer, we have the criterion that to qualify, a graph  $G$  must contain a  $d$ -clique of suspensions such that there is an orientation of the edges of  $G$  with  $\text{out-deg}(x) = d$  for all non-suspensions  $x$ .

**Problem 4** Identify additional obstructions against having a  $d$ -realizer.

Another situation where induced subgraphs of graphs with a  $(d+1)$ -realizer appear is given by families of interiorly disjoint pairwise homothetic  $d$ -simplices in  $d$ -space with vertex-facet incidences. To produce a  $d$ -realizer for a supergraph add a small  $d$ -simplex over each vertex that does not take part in a vertex-facet contact and then use the directions of inward pointing normals of the facets to list the simplices. Figure 5 shows an example in 2 dimensions.

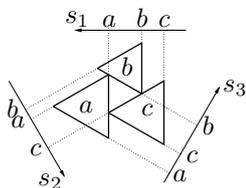


Fig. 5: A 3-realizer from homothetic triangles.

**Problem 5** Is it possible to realize every simple  $d$ -tree as vertex-facet contact graph of homothetic simplices in  $\mathbb{R}^d$ ?

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