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Functional analysis

Solutions to 1. Exercise Sheet

Exercise 1 ((C) Extension principle)

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two normed spaces and furthermore assume that Y is complete. Now let $A \subseteq X$ be a dense subset of X and $f: A \to Y$ a function. Show that, if the function f is uniformly continuous, then there is an unique continuous function $\widetilde{f}: X \to Y$ such that $\widetilde{f}|_A = f$ holds. Proof with a counterexample the necessity of the uniform continuity.

Solution of Exercise 1

By the definition of the uniform continuity for every $\varepsilon > 0$ we find a $\delta(\varepsilon) =: \delta > 0$ such that we have

$$\|f(x) - f(y)\|_Y < \frac{\varepsilon}{3} \text{ for every } x, y \in X \text{ with } \|x - y\|_X < \delta.$$

Let $\varepsilon > 0$ be arbitrary. If $(x_n)_{n\in\mathbb{N}} \subseteq A$ is a converging sequence in X, i.e. we find an unique element $x \in X$ such that

$$\|x_n - x\|_X \to 0 \text{ as } n \to \infty,$$

then $(x_n)_{n\in\mathbb{N}}$ is especially a Cauchy-sequence in X, i.e. we can choose $n_0(\varepsilon) =: n_0 \in \mathbb{N}$ with

$$\|x_n - x_m\| < \delta \text{ for every } n, m \in \mathbb{N} \text{ with } n, m \geq n_0.$$

Now we get

$$\|f(x_n) - f(x_m)\|_Y < \frac{\varepsilon}{3} < \varepsilon \text{ for every } n, m \in \mathbb{N} \text{ with } n, m \geq n_0.$$

Since ε was arbitrary, the sequence $(f(x_n))_{n\in\mathbb{N}} \subseteq Y$ is a Cauchy-sequence in Y. Since the space Y is complete we know by definition that the sequence $(f(x_n))_{n\in\mathbb{N}}$ is converging to an unique element $y_{(x_n)_n} \in Y$ in Y.
Let $(z_n)_{n\in\mathbb{N}} \subseteq A$ be another sequence which converge to x in X. Then we know by the previous argument that the sequence $(f(z_n))_{n\in\mathbb{N}}$ converge to an unique element $y_{(z_n)_n} \in Y$ in Y. Now choose $n_0(\varepsilon) =: n_0 \in \mathbb{N}$ such that

$$\left\|y_{(x_n)_n} - f(x_n)\right\|_Y < \frac{\varepsilon}{3}, \ \left\|y_{(z_n)_n} - f(z_n)\right\|_Y < \frac{\varepsilon}{3},$$
$$\|x_n - x\|_X < \frac{\delta}{2}, \ \|z_n - x\|_X < \frac{\delta}{2}$$

holds for every $n \in \mathbb{N}$ with $n \geq n_0$. Then we have by two times triangle inequality and since $\|x_n - z_n\|_X \leq \|x_n - x\|_X + \|x - z_n\|_X < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ holds that

$$\begin{aligned}\left\|y_{(x_n)_n} - y_{(z_n)_n}\right\|_Y &\leq \left\|y_{(x_n)_n} - f(x_n)\right\|_Y + \left\|f(x_n) - y_{(z_n)_n}\right\|_Y \\ &\leq \left\|y_{(x_n)_n} - f(x_n)\right\|_Y + \|f(x_n) - f(z_n)\|_Y + \left\|f(z_n) - y_{(z_n)_n}\right\|_Y \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

i.e. $y_x := y_{(x_n)_n} = y_{(z_n)_n} \in Y$ in Y because ε was arbitrary.
Now we can define a well-defined function $\widetilde{f}: X \to Y$ as $\widetilde{f}(x) = y_x$ for $x \in X$ because A is dense in X. So $\widetilde{f}|_A = f$ since for $x \in A$ we choose $x_n = x \in A$ for $n \in \mathbb{N}$ and so we have $f(x_n) = f(x) \to f(x)$ in Y as $n \to \infty$, i.e. $\widetilde{f}(x) = f(x)$.
Now let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a converging sequence to x in X. Choose by density of A in X for every x_n, $n \in \mathbb{N}$, and for x a sequence $\left(z_m^{(n)}\right)_{m\in\mathbb{N}} \subseteq A$ resp. $(z_n)_{n\in\mathbb{N}} \subseteq A$ which converge to x_n resp. x in X. Then choose an index $n_0(\varepsilon, \delta) =: n_0 \in \mathbb{N}$ such that

$$\left\|f(z_n) - \widetilde{f}(x)\right\|_Y < \frac{\varepsilon}{3},$$
$$\|x_n - x\|_X < \frac{\delta}{3}, \ \|z_n - x\|_X < \frac{\delta}{3}$$

for every $n \in \mathbb{N}$ with $n \geq n_0$ holds. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Then we can find an index $m_0 := m_0(\varepsilon, n_0, \delta) \in \mathbb{N}$ with

$$\left\|\widetilde{f}\left(x_n\right) - f\left(z_m^{(n)}\right)\right\|_Y < \frac{\varepsilon}{3},$$
$$\left\|z_m^{(n)} - x_n\right\|_X < \frac{\delta}{3}$$

for all $m \in \mathbb{N}$ with $m \geq m_0$. It follows directly by two times triangle inequality for all $m \in \mathbb{N}$ with $m \geq m_0$:

$$\begin{aligned}\left\|z_m^{(n)} - z_n\right\|_X &\leq \left\|z_m^{(n)} - x_n\right\|_X + \|x_n - z_n\|_X \\ &\leq \left\|z_m^{(n)} - x_n\right\|_X + \|x_n - x\|_X + \|x - z_n\|_X \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \\ \text{i.e. } \left\|f\left(z_m^{(n)}\right) - f\left(z_n\right)\right\|_Y &< \frac{\varepsilon}{3}.\end{aligned}$$

We conclude by two times triangle inequality that for $m \in \mathbb{N}$ with $m \geq m_0$ hold

$$\begin{aligned}\left\|\widetilde{f}\left(x_n\right) - \widetilde{f}(x)\right\|_Y &\leq \left\|\widetilde{f}\left(x_n\right) - f\left(z_n\right)\right\|_Y + \left\|f\left(z_n\right) - \widetilde{f}(x)\right\|_Y \\ &\leq \left\|\widetilde{f}\left(x_n\right) - f\left(z_m^{(n)}\right)\right\|_Y + \left\|f\left(z_m^{(n)}\right) - f\left(z_n\right)\right\|_Y + \left\|f\left(z_n\right) - \widetilde{f}(x)\right\|_Y \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

i.e. $\widetilde{f}$ is continuous on X ($\widetilde{f} \in C^0(X, Y)$).
If $g\colon X \to Y$ is another continuous function on X with $g|_A = f$, then we have for any $x \in X$ and any sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ which converge to x in X (by density of A in X):

$$\begin{aligned}\left\|\widetilde{f}(x) - g(x)\right\|_Y &\leq \left\|\widetilde{f}(x) - f\left(x_n\right)\right\|_Y + \left\|f\left(x_n\right) - g(x)\right\|_Y \\ &= \left\|\widetilde{f}(x) - \widetilde{f}\left(x_n\right)\right\|_Y + \left\|g\left(x_n\right) - g(x)\right\|_Y \to 0 \text{ as } n \to \infty,\end{aligned}$$

i.e. $\widetilde{f}(x) = g(x)$ for every $x \in X$ and this is equivalent to $\widetilde{f} = g$ on X. This means that the continuous function $\widetilde{f}$ is unique. □

Counterexample: Set $X = Y = \mathbb{R}$ with $\|\cdot\|_X = \|\cdot\|_Y = |\cdot|$, $A = \mathbb{R}\backslash\{0\} \subseteq \mathbb{R}$ and the function

$$f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R},\ x \mapsto \frac{1}{x^2}.$$

It is clear that $(\mathbb{R}, |\cdot|)$ is a complete normed space, i.e. a Banach space, and that the function f is continuous on $\mathbb{R}\backslash\{0\}$, and $\mathbb{R}\backslash\{0\}$ is dense in $\mathbb{R}$. But the function f has no continuous extension in zero. This shows that it is important to have uniform continuity. □

Exercise 2 (Metric spaces)

1. Let (X, d_X) be a metric space.

   (M1) Show: If $h\colon X \to X$ is an injective map, then $d_h(x, y) := d_X(h(x), h(y))$ is a metric on X.

   (M2) Now we set $X := \mathbb{R}$ with the standard metric (=absolute value). Is the metric space $(\mathbb{R}, d_{\arctan})$ complete?

2. Proof: We have via

   $$\|u\|_{L^2([-1,1])} := \left(\int_{-1}^{1} |u(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

   a norm on $C^0([-1, 1], \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, but the space $\left(C^0([-1, 1], \mathbb{K}), \|\cdot\|_{L^2([-1,1])}\right)$ is not a Banach space.

Exercise 3 ((C) Banach spaces)

Let $(X, \|\cdot\|_X)$ be a normed space. Show the following:

(C1) X is a Banach space if and only if for all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $\sum\limits_{n=1}^{\infty} \|x_n\|_X < \infty$ there exists an element $x \in X$ with

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\|_X = 0.$$

(C2) We define the sequence-space l^p for $p \in [1, \infty]$ via

$$l^p := \left\{ x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} \colon \|x\|_{l^p} := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\} \text{ if } p < \infty,$$
$$l^\infty := \left\{ x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} \colon \|x\|_{l^\infty} := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that for every $p \in [1, \infty]$ the space $(l^p, \|\cdot\|_{l^p})$ is a Banach space.

Exercise 4 ()

Let (X, d_X) be a metric space and let $Y \subseteq X$ be a subset of X. Show the following:

(D1) If (Y, d_X) is complete, then Y is closed in X.

(D2) If (X, d_X) is complete and Y is closed in X, then (Y, d_X) is also complete.

Solution of Exercise 4

Without loss of generality (W.l.o.g) we assume that Y is not empty ($Y \neq \emptyset$), since otherwise the statements are clear by definition.
(D1) Let (Y, d_X) be complete and assume that $(x_n)_{n\in\mathbb{N}} \subseteq Y$ is a converging sequence to an Element $x \in X$ in X. Then we know autamically that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in Y. The completness of Y implies that the sequence $(x_n)_{n\in\mathbb{N}}$ has to converge in Y to a limit in Y. By the uniqueness of limits and $Y \subseteq X$ we get that

$$y = \lim_{n\to\infty} x_n \in Y,$$

i.e. Y is closed in X since the converging sequence $(x_n)_{n\in\mathbb{N}}$ was arbitrary. □
(D2) Let (X, d_X) be complete and Y be closed in X. Let $(x_n)_{n\in\mathbb{N}} \subseteq Y$ be an arbitrary Cauchy-sequence in Y, then $(x_n)_{n\in\mathbb{N}}$ is especially a Cauchy-sequence in X, since $Y \subseteq X$. Since X is complete the Cauchy-sequence $(x_n)_{n\in\mathbb{N}}$ has to converge in X with limit $x \in X$. The converging sequence $(x_n)_{n\in\mathbb{N}}$ is in Y and Y is closed in X and so the limit x is in Y, i.e. $x \in Y$. The Cauchy-sequence $(x_n)_{n\in\mathbb{N}}$ was arbitrary and so we get that Y is also complete. □