

Functional analysis

2. Exercise Sheet

Exercise 1 (Cartesian Product of Banach spaces)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and we define the map

$$\|\cdot\|_{X \times Y} : X \times Y \rightarrow \mathbb{R}, (x, y) \mapsto \max\{\|x\|_X, \|y\|_Y\}.$$

1. Show that $(X \times Y, \|\cdot\|_{X \times Y})$ is a normed space.
2. Show that if X and Y are Banach spaces, then $(X \times Y, \|\cdot\|_{X \times Y})$ is also a Banach space.

Exercise 2 ((C) Multiplication Operators)

1. Let $\Omega \subseteq \mathbb{R}^d$, $X := C_b^0(\Omega, \mathbb{K}) := \{f: \Omega \rightarrow \mathbb{K} : f \text{ is bounded and continuous on } \Omega\}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in X$ and M be the Multiplication operator given by

$$M: X \rightarrow X, Mf = m \cdot f.$$

Show that M is well-defined with operator norm $\|M\| = \|m\|_{C^0(\Omega, \mathbb{K})}$.

2. Let $(\Omega, \mathcal{A}, \mu)$ a σ -finite measure space and $m: \Omega \rightarrow \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be μ -measurable, $p \in [1, \infty]$. We define formally the multiplication operator M by

$$Mf = m \cdot f.$$

Show the following:

$$M \in B(L^p(\Omega, \mu), L^p(\Omega, \mu)) \Leftrightarrow m \in L^\infty(\Omega, \mu).$$

In this case we have for the operator norm of M :

$$\|M\| = \|m\|_{L^\infty(\Omega, \mu)}.$$

Exercise 3 ((C) Integral operator)

Set $I := [0, 1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T: (C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I, \mathbb{K})}) \rightarrow (C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I, \mathbb{K})})$ with kernel k by

$$Tf(x) = \int_0^1 k(x, y)f(y)dy, \quad x \in I.$$

Show that the integral operator T is well-defined and compute the operator norm $\|T\|$.

Exercise 4 (Finite dimensional subspaces)

Let $(X, \|\cdot\|_X)$ be a normed space and $U \subseteq X$ be a finite dimensional subspace. Show that U is closed in X .

Exercise Sheets

There will be one exercise sheet every week with three or four problems which you can find on the webpage. You can hand in your solution of the two exercises which are labeled with a 'C' and then the tutor will correct it. Please deliver your solution in the box on the ground floor of the math building or in the problem class. In the problem class we will discuss the solutions. If I have not enough time, I will upload my sketches to the missing problems on the webpage.