10.11.2016
Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Karlsruhe Institute of Technology

Functional analysis

4. Exercise Sheet

Exercise 1 (Application of the Hahn-Banach Theorem)

Find a functional $\Phi \in l^\infty(\mathbb{N},\mathbb{R})'$ with $\|\Phi\| = 1$ and the following three properties:

(1) $\Phi(x) = \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ for $x \in l^\infty(\mathbb{N},\mathbb{R})$, if this limit exists.

(2) $\liminf\limits_{i\to\infty} x_i \leq \Phi(x) \leq \limsup\limits_{i\to\infty} x_i$ for $x \in l^\infty(\mathbb{N},\mathbb{R})$.

(3) $\Phi\left((x_1,x_2,x_3,\ldots)\right) = \Phi\left((x_2,x_3,\ldots)\right)$ for $x = (x_1,x_2,x_3,\ldots) \in l^\infty(\mathbb{N},\mathbb{R})$.

Exercise 2 ((C) Projection to finite-dimensional subspaces)

Let $(X, \|\cdot\|_X)$ be a normed space and $U \subseteq X$ be a finite-dimensional subspace of X. We call a linear map $P\colon X \to X$ a Projection if and only if $P^2 = P$ on X. Construct a continuous Projection P with $\mathrm{Im}(P) = U$ and conclude that the equality

$$X = \ker(P) \oplus U$$

holds.

Exercise 3 ((C) Separability)

Show that every subset A of a separable metric space (X, d_X) is also separable (with the induced metric). Conclude that the space $(l^\infty(\mathbb{N},\mathbb{R}), \|\cdot\|_{l^\infty})$ is not separable.

Exercise 4 (Product of Hölder-continuous functions)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with some chord-arc condition and $k \in \mathbb{N}_0, \alpha \in (0,1]$ be arbitrary. Show that if $u, v \in C^{k,\alpha}\left(\overline{\Omega}\right)$ then the product $u \cdot v$ is also in $C^{k,\alpha}\left(\overline{\Omega}\right)$ with norm estimate

$$\|u \cdot v\|_{C^{k,\alpha}(\Omega)} \leq C\left(k,\Omega\right) \|u\|_{C^{k,\alpha}(\Omega)} \|v\|_{C^{k,\alpha}(\Omega)} \text{ for some constant } C\left(k,\Omega\right) > 0.$$