17.11.2016
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Functional analysis

5. Exercise Sheet

Exercise 1 ((C) About Bounded Inverse)

Find an example for a continuous operator $T \in L(X, X)$ on a normed space $(X, \|\cdot\|_X)$ such that the inverse T^{-1} is not continuous.

Exercise 2 (Continuity of bilinear forms)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces and $B\colon X \times Y \to \mathbb{R}$ be a bilinear form such that for every $x \in X$ resp. $y \in Y$ the map $B(x, \cdot)$ resp. $B(\cdot, y)$ is continuous. Show that B is continuous on $\left(X \times Y, \|\cdot\|_{X\times Y}\right)$ with $\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$ for $(x, y) \in X \times Y$.

Exercise 3 ((C) Pointwise convergent operators)

Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ a normed space and $(T_n)_{n\in\mathbb{N}} \subseteq L(X, Y)$ be a pointwise convergent operator sequence, i.e. for every $x \in X$ there is an element $Tx \in Y$ with $Tx = \lim\limits_{n\to\infty} T_n x$ in Y.

(1) Show that the operator $T \in L(X, Y)$ and $\|T\| \leq \liminf\limits_{n\to\infty} \|T_n\|$.

(2) Find an example for "$<$" in the inequality in (1).

Exercise 4 (Commutators)

Let $(X, \|\cdot\|_X)$ be a normed space and let $P, Q\colon X \to X$ be linear with $[P, Q] := PQ - QP = \mathrm{Id}_X$. Show that either P or Q is not continuous on X.
(Hint: Show first that $PQ^{n+1} - Q^{n+1}P = (n + 1)Q^n$ holds for all $n \in \mathbb{N}$.)