01.12.2016
Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Functional analysis

Solutions to 7. Exercise Sheet

Exercise 1 ((C) Dual spaces of c_0 and c)

What are the dual spaces of

$$c_0 := \left\{ x = (x_1, x_2, x_3, \dots) \in l^\infty(\mathbb{N}, \mathbb{R}) : \lim_{n\to\infty} x_n = 0 \right\},$$
$$c := \left\{ x = (x_1, x_2, x_3, \dots) \in l^\infty(\mathbb{N}, \mathbb{R}) : \lim_{n\to\infty} x_n \text{ exists in } \mathbb{R} \right\}$$

endowed with the sup-norm $\|\cdot\|_{l^\infty(\mathbb{N})}$?

Exercise 2 ((C) Characterisation of strictly normed $\mathbb{R}$-spaces)

Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{R}$-vectorspace. Show that the following are equivalent:

(1) X is strictly normed, i.e. for all $x, y \in X$ we have

$$\|x+y\|_X = \|x\|_X + \|y\|_X \ \Rightarrow\ x, y \text{ are linearly dependent.}$$

(2) The unit ball $B_1(0) \subseteq X$ is strictly convex, i.e. for all $x, y \in X$ we have

$$\|x\|_X = \|y\|_X = 1,\ x \neq y \ \Rightarrow\ \left\| \frac{x+y}{2} \right\|_X < 1.$$

(3) Every closed convex subset $K \subseteq X$ has at most one element with minimal norm.

Solution of Exercise 2

(1)⇒(2): Let $x, y \in X$ with $\|x\|_X = 1 = \|y\|_X$ and $x \neq y$ in X. If we assume that $\left\|\frac{x+y}{2}\right\|_X \geq 1$ we get

$$2 \leq \|x+y\|_X \leq \|x\|_X + \|y\|_X = 1 + 1 = 2, \text{ i.e. } \|x+y\|_X = 2 = \|x\|_X + \|y\|_X .$$

The property (1) implies now that x and y are linear dependent, so we find some number $\lambda \in \mathbb{K}$ such that $x = \lambda y$. But then we have that

$$2 = \|x+y\|_X = \|x + \lambda x\|_X = |1+\lambda| \, \|x\|_X , \text{ and } 1 = \|y\|_X = \|\lambda x\|_X = |\lambda| \, \|x\|_X = |\lambda| ,$$

i.e. $\lambda = 1$. Then we get that $x = y$, which is a contradiction to $x \neq y$, i.e. our assumption was wrong and (2) holds.
(2)⇒(3): Let $K \subseteq X$ be a closed and convex subset of X and we assume w.l.o.g. that there is some element in K with minimal norm. Let $y, z \in K$ be two elements with minimal norm in K and set

$$r := \inf_{x\in K} \|x\|_X , \text{ i.e. } \|y\|_X = \|z\|_X = r.$$

If $r = 0$, then $\|y\|_X = \|z\|_X = 0$ and that means $y = z = 0$ in X. Hence let $r > 0$. Since K is convex we know that $\frac{y+z}{2} \in K$, and we have for the norm:

$$\left\| \frac{y+z}{2} \right\|_X \geq r.$$

If $y \neq z$ in X, we can apply property (2) for $\frac{y}{r}$ and for $\frac{z}{r}$ and we get

$$1 > \left\| \frac{\frac{y}{r} + \frac{z}{r}}{2} \right\|_X = \left\| \frac{y+z}{2r} \right\|_X = \frac{1}{r} \left\| \frac{y+z}{2} \right\|_X \ \Leftrightarrow\ r > \left\| \frac{y+z}{2} \right\|_X ,$$

which is a contradiction to

$$\left\| \frac{y+z}{2} \right\|_X \geq r,$$

i.e. assumption was wrong and $y = z$ in X, i.e. (3) holds.
(3)⇒(1): Let $x, y \in X$ be linearly independent, especially $x \neq 0 \neq y$. We take the normed elements:

$$\widetilde{x} := \frac{x}{\|x\|_X} \text{ and } \widetilde{y} := \frac{y}{\|y\|_X}.$$

And we define the subset

$$K := \{(1-s)\widetilde{x} + s\widetilde{y} \colon s \in [0,1]\} \subseteq X.$$

As a set of all convex-combinations of $\widetilde{x}, \widetilde{y}$ the subset K is convex. Let $(z_n)_{n\in\mathbb{N}} \subseteq K$ be a converging sequence to an element $z \in X$ in X. Then we find for every $n \in \mathbb{N}$ some $s_n \in [0,1]$ such that

$$z_n = (1-s_n)\,\widetilde{x} + s_n\widetilde{y}.$$

Since the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded we get with the theorem of Bolzano-Weierstraß a converging subsequence $(s_{n_k})_{k\in\mathbb{N}}$ to some $s \in \mathbb{R}$ of $(s_n)_{n\in\mathbb{N}}$. The interval $[0,1]$ is compact, especially closed, and so $s \in [0,1]$, and that implies that

$$z = \lim_{n\to\infty} z_n = \lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} \left[(1-s_{n_k})\,\widetilde{x} + s_{n_k}\widetilde{y}\right] = (1-s)\widetilde{x} + s\widetilde{y} \in K,$$

i.e. K is closed. Now we take the functional $\varphi \colon [0,1] \to [0,\infty),\ s \mapsto \|(1-s)\widetilde{x} + s\widetilde{y}\|_X$. This functional is continuous as a composition of continuous function and φ is convex, since for some $a, b, t \in [0,1]$ we have that

$$\begin{aligned}
\varphi\,(ta + (1-t)b) &= \|(1 - ta - (1-t)b)\,\widetilde{x} + (ta + (1-t)b)\,\widetilde{y}\|_X \\
&= \|t\,(1-a)\,\widetilde{x} + ta\widetilde{y} + (1-t)\,(1-b)\,\widetilde{x} + (1-t)b\widetilde{y}\|_X \\
&\leq t\,\|(1-a)\widetilde{x} + a\widetilde{y}\|_X + (1-t)\,\|(1-b)\widetilde{x} + b\widetilde{y}\|_X \\
&= t\varphi(a) + (1-t)\varphi(b).
\end{aligned}$$

Since φ is continuous on $[0,1]$ and $[0,1]$ is compact in $\mathbb{R}$ there exists a point $s_0 \in [0,1]$ with $\varphi\,(s_0) = \min_{s\in[0,1]} \varphi(s) \geq 0$, i.e. in K there is some element $((1-s_0)\,\widetilde{x} + s_0\widetilde{y} \in K)$ with minimal norm. Property (3) gives us that this element is unique, and since

$$\varphi(0) = \|(1-1)\widetilde{x} + 1\widetilde{y}\|_X = \|\widetilde{y}\|_X = 1 = \|\widetilde{x}\|_X = \|(1-0)\widetilde{x} + 0\widetilde{y}\|_X = \varphi(0)$$

we know that $s_0 \in (0,1)$ and $\varphi\,(s_0) \in [0,1)$. In general we have by triangle inequality:

$$0 \leq \varphi(s) = \|(1-s)\widetilde{x} + s\widetilde{y}\|_X \leq (1-s)\,\|\widetilde{x}\|_X + s\,\|\widetilde{y}\|_X = (1-s) + s = 1 \text{ for all } s \in [0,1].$$

And for some $s' \in (0,1)$ there are $t, s^* \in [0,1]$ such that $s = (1-t)s_0 + ts^*$ then we get since φ is convex

$$\varphi(s) = \varphi\,((1-t)s_0 + ts^*) \leq (1-t)\varphi\,(s_0) + t\varphi\,(s^*) \leq (1-t)\varphi\,(s_0) + t \cdot 1 < (1-t) \cdot 1 + t = 1,$$

i.e. $\varphi < 1$ on $(0,1)$. We set $r := \frac{\|x\|_X + \|y\|_X}{2} > 0$ and

$$0 < s := \frac{\|y\|_X}{2r} = \frac{\|y\|_X}{\|x\|_X + \|y\|_X} < 1.$$

Then we have that

$$\begin{aligned}
r\,((1-s)\widetilde{x} + s\widetilde{y}) &= r\left(\left(1 - \frac{\|y\|_X}{2r}\right)\widetilde{x} + \frac{\|y\|_X}{2r} \cdot \frac{y}{\|y\|_X}\right) \\
&= r\left(\frac{2r - \|y\|_X}{2r} \cdot \frac{x}{\|x\|_X} + \frac{y}{2r}\right) \\
&= \frac{\|x\|_X + \|y\|_X - \|y\|_X}{2} \cdot \frac{x}{\|x\|_X} + \frac{y}{2r} = \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2}.
\end{aligned}$$

This implies that

$$1 > \varphi(s) = \left\| \frac{x+y}{2r} \right\|_X \;\Leftrightarrow\; \|x+y\|_X < 2r = \|x\|_X + \|y\|_X\,,$$

i.e. (1) holds.
This proves the equivalence. □

Exercise 3 (Hausdorff measures are Borel-regular)

Show that the Hausdorff measure $\mathcal{H}^s$, $s \geq 0$, on a metric space (X, d_X) is a Borel-regular outer measure where we define for all subsets $A \subseteq X$ and $\delta > 0$ the outer measures

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{n=1}^{\infty} \operatorname{diam}(A_n)^s : A \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and } \operatorname{diam}(A_n) \leq \delta \text{ for all } n \in \mathbb{N}\right\}, \ \inf \emptyset := \infty,$$
$$\mathcal{H}^s(A) := \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Solution of Exercise 3

(4) $\mathcal{H}^s$ is Borel regular.
For any $C \subseteq X$ we have that $\operatorname{diam}(C) = \operatorname{diam}\left(\overline{C}\right)$, since for $x, y \in \overline{C}$ we can choose sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq C$ with $x_n \to x$ and $y_n \to y$ in X, then we have

$$d_X(x,y) = \lim_{n\to\infty} d_X(x_n, y_n) \leq \lim_{n\to\infty} \operatorname{diam}(C) = \operatorname{diam}(C),$$

i.e.

$$\operatorname{diam}(C) \leq \operatorname{diam}\left(\overline{C}\right) \leq \operatorname{diam}(C).$$

Therefore we can also write

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{n=1}^{\infty} \operatorname{diam}(A_n)^s : A \subseteq \bigcup_{n=1}^{\infty} A_n, \ \operatorname{diam}(A_n) \leq \delta, \ A_n \subseteq X \text{ is closed}\right\}$$

for all $A \subseteq X$. Let $A \subseteq X$ with $\mathcal{H}^s(A) < \infty$, otherwise it is trivial, since X is a Borel set with $\mathcal{H}^s(X) = \infty$. We know that $\mathcal{H}^s_\delta(A) < \infty$ for all $\delta > 0$. For every $k \in \mathbb{N}$ choose a sequence $\left(C_n^{(k)}\right)_{n\in\mathbb{N}} \subseteq X$ such that $\operatorname{diam}\left(C_n^{(k)}\right) \leq \frac{1}{k}$ with $A \subseteq \bigcup_{n=1}^{\infty} C_n^{(k)}$ and with

$$\sum_{n=1}^{\infty} \operatorname{diam}\left(C_n^{(k)}\right) \leq \mathcal{H}^s_{\frac{1}{k}}(A) + \frac{1}{k}.$$

We set for $k \in \mathbb{N}$

$$A_k := \bigcup_{n=1}^{\infty} C_n^{(k)}, \ B := \bigcap_{k=1}^{\infty} A_k.$$

Then the subset B is Borel (by definition) and $A \subseteq A_k$ for all $k \in \mathbb{N}$, i.e. $A \subseteq B$. We get that

$$\mathcal{H}^s_{\frac{1}{k}}(B) \leq \sum_{n=1}^{\infty} \operatorname{diam}\left(C_n^{(k)}\right) \leq \mathcal{H}^s_{\frac{1}{k}}(A) + \frac{1}{k} \text{ for all } k \in \mathbb{N}.$$

For $k \to \infty$ we get now

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A) \leq \mathcal{H}^s(B), \text{ since } A \subseteq B,$$

i.e.

$$\mathcal{H}^s(A) = \mathcal{H}^s(B).$$

This means that the Borel outer measure $\mathcal{H}^s$ is Borel regular for all $s \geq 0$. □

Exercise 4 ($C_c^0(X, \mathbb{R})$ is dense in $L^1(X, \mu)$)

Let (X, d_X) be a σ-compact metric space. Show that the space $C_c^0(X, \mathbb{R})$ is dense in $L^1(X, \mu)$ for all Radon measures μ on X.
(Hint: Use the fact that step-functions are dense in $L^1(X, \mu)$.)

Solution of Exercise 4

Step 01.: Nonnegative functions $f \in L^1(X, \mu)$ can be approximated by stepfunctions.
Let $f \in L^1(X, \mu)$ be a nonnegative function, i.e. $f \geq 0$ on X. We define for all $k, n \in \mathbb{N}_0$ the set

$$A_{k,n} := \left\{x \in X : 2^{-n}k \leq f(x) < 2^{-n}(k+1)\right\},$$

and the stepfunctions

$$s_n := \sum_{k=0}^{2^{2n}} 2^{-n} k \chi_{A_{k,n}} \geq 0 \text{ on } X,$$

where the characteristic function $\chi_B \colon X \to \{0,1\}$ for some subset $B \subseteq X$ is defined by

$$\chi_B(x) := \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases} \quad \text{for } x \in X.$$

It is for $n \in \mathbb{N}_0$

$$B_n := \dot{\bigcup_{k=0}^{2^{2n}}} A_{k,n} = \left\{ x \in X \colon 0 \leq f(x) < 2^n + 2^{-n} \right\}, \; X = \bigcup_{n=1}^{\infty} B_n.$$

If we take some point $x \in B_n$, $n \in \mathbb{N}_0$, then we find some unique $k_0 \in \mathbb{N}_0$ such that $x \in A_{k_0,n}$, it follows:

$$\begin{aligned} |f(x) - s_n(x)| &= \left| f(x) - \sum_{k=0}^{2^{2n}} 2^{-n} k \chi_{A_{k,n}}(x) \right| \\ &= \left| f(x) - 2^{-n} k_0 \chi_{A_{k_0,n}}(x) \right| = \left| f(x) - 2^{-n} k_0 \right| \\ &= f(x) - 2^{-n} k_0 < 2^{-n} (k_0 + 1) - 2^{-n} k_0 = 2^{-n} \to 0 \text{ for } n \to \infty, \end{aligned}$$

i.e. $(s_n)_{n \in \mathbb{N}}$ converge pointwise to f on X. Since the function $f \in L^1(X, \mu)$, we know that $s_n \in L^1(X, \mu)$ for $n \in \mathbb{N}$, because

$$\begin{aligned} |s_n(x)| &= \sum_{k=0}^{2^{2n}} 2^{-n} k \chi_{A_{k,n}} < \sum_{k=0}^{2^{2n}} f(x) \chi_{A_{k,n}} \\ &= f(x) \chi_{\bigcup_{k=0}^{2^{2n}}}(x) = f(x) \chi_{B_n}(x) \\ \|s_n\|_{L^1(X,\mu)} &= \int_X |s_n(x)| \, \mathrm{d}\mu(x) < \int_X f(x) \chi_{B_n}(x) \mathrm{d}\mu(x) = \int_{B_n} f(x) \mathrm{d}\mu(x) \\ &\leq \int_X |f(x)| \, \mathrm{d}\mu(x) = \|f\|_{L^1(X,\mu)} < \infty. \end{aligned}$$

The Theorem of Lebesgue gives us that the sequence $(s_n)_{n \in \mathbb{N}}$ converge to the function f in $L^1(X, \mu)$, i.e.

$$\lim_{n \to \infty} \|f - s_n\|_{L^1(X,\mu)} = \lim_{n \to \infty} \int_X |f(x) - s_n(x)| \, \mathrm{d}\mu(x) = \lim_{n \to \infty} \int_X (f(x) - s_n(x)) \, \mathrm{d}\mu(x) = 0.$$

Step 02.: Functions $f \in L^1(X, \mu)$ can be approximated by stepfunctions.
We decomposite f in positive and negative part, i.e. $f = f^+ - f^-$ with

$$f^+(x) := \sup \{0, f(x)\}, \; f^-(x) := \sup \{0, -f(x)\} \text{ for } x \in X.$$

Then we have that $f^+, f^- \geq 0$ on X and $f^+, f^- \in L^1(X, \mu)$, since we have

$$\left\| f^\pm \right\|_{L^1(X,\mu)} = \int_X \left| f^\pm(x) \right| \mathrm{d}\mu(x) \leq \int_X |f(x)| \, \mathrm{d}\mu(x) = \|f\|_{L^1(X,\mu)} < \infty.$$

By step 01. there are two sequences of stepfunctions $\left(s_n^{(1)} \right)_{n \in \mathbb{N}}$, $\left(s_n^{(2)} \right)_{n \in \mathbb{N}} \subseteq L^1(X, \mu)$ such that

$$\lim_{n \to \infty} \left\| f^+ - s_n^{(1)} \right\|_{L^1(X,\mu)} = 0, \; \lim_{n \to \infty} \left\| f^- - s_n^{(2)} \right\|_{L^1(X,\mu)} = 0.$$

This implies by triangle inequality that

$$\begin{aligned} \lim_{n \to \infty} \left\| f - \left(s_n^{(1)} - s_n^{(2)} \right) \right\|_{L^1(X,\mu)} &= \lim_{n \to \infty} \left\| f^+ - f^- - \left(s_n^{(1)} - s_n^{(2)} \right) \right\|_{L^1(X,\mu)} \\ &\leq \lim_{n \to \infty} \left(\left\| f^+ - s_n^{(1)} \right\|_{L^1(X,\mu)} + \left\| s_n^{(2)} - f^- \right\|_{L^1(X,\mu)} \right) \\ &= \lim_{n \to \infty} \left\| f^+ - s_n^{(1)} \right\|_{L^1(X,\mu)} + \lim_{n \to \infty} \left\| s_n^{(2)} - f^- \right\|_{L^1(X,\mu)} \\ &= 0 + 0 = 0, \end{aligned}$$

i.e. the function f can be approximated by stepfunctions.
Step 03.: Stepfunctions can be approximated by continuous functions with compact support.
Let $A \subseteq X$ be a measurable subset of X with $\mu(A) < \infty$ and set $f = \chi_A$ on X. By a conclusion of the lecture we know that

$$\mu(A) = \sup_{K \subseteq A,\ K \text{ compact}} \mu(K).$$

Let be $\varepsilon > 0$. So we can choose some compact subset $K \subseteq A$ such that

$$\mu(K) > \mu(A) - \varepsilon,$$

i.e.

$$\mu(A \backslash K) = \mu(A) - \mu(K) < \mu(A) - (\mu(A) - \varepsilon) = \varepsilon.$$

Define the sequence

$$f_n \colon \to [0,1],\ x \mapsto f_n(x) := \left(1 - \frac{\operatorname{dist}(x,K)}{n}\right)^+ := \sup\left\{0, 1 - \frac{\operatorname{dist}(x,K)}{n}\right\} \text{ for } n \in \mathbb{N}.$$

Then we have $f_n \in C^0(X)$ with support

$$\operatorname{supp}(f_n) = \{x \in X \colon \operatorname{dist}(x,K) \leq n\} \text{ for } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ the function f_n has compact support, since for $\delta > 0$ we find finite many point $x_1, \ldots, x_k \in K$, $k \in \mathbb{N}$, with

$$K \subseteq \bigcup_{i=1}^{k} B_\delta(x_i),$$

because the subset K is compact. Then we get that

$$\operatorname{supp}(f_n) \subseteq \overline{\bigcup_{i=1}^{k} B_{\delta+n}(x_i)} = \bigcup_{i=1}^{k} \overline{B_{\delta+n}(x_i)},$$

and the union $\bigcup_{i=1}^{k} \overline{B_{\delta+n}(x_i)}$ is compact as a finite union of compact subsets, because of the σ-compactness the closed balls $\overline{B_{\delta+n}(x_i)}$ for $i = 1, \ldots, k$ are compact. Since the support of f_n is a closed subset in a compact set we know that $\operatorname{supp}(f_n)$ is compact, i.e. $f_n \in C_c^0(X)$. We have $f_n = f$ on K for all $n \in \mathbb{N}$, then it follows:

$$\begin{aligned} \|f_n - f\|_{L^1(X,\mu)} &= \int_X |f_n(x) - \chi_A(x)| \, \mathrm{d}\mu(x) = \int_X (\chi_A(x) - f_n(x)) \, \mathrm{d}\mu(x) \\ &\leq \int_X (\chi_A(x) - \chi_K(x)) \, \mathrm{d}\mu(x) = \int_X \chi_{A \backslash K}(x) \mathrm{d}\mu(x) \\ &= \mu(A \backslash K) < \varepsilon, \end{aligned}$$

i.e. f can be approximated by continuous functions with compact support.
Step 04.: Functions in $L^1(X,\mu)$ can be approximated by continuous functions with compact support.
Let $f \in L^1(X,\mu)$ be a function. Then by step 02. there is a sequence $(s_n)_{n \in \mathbb{N}} \subseteq L^1(X,\mu)$ of stepfunctions such that

$$\lim_{n \to \infty} \|f - s_n\|_{L^1(X,\mu)} = 0.$$

For every $n \in \mathbb{N}$ by step 03. there is a sequence $\left(g_k^{(n)}\right)_{k \in \mathbb{N}} \subseteq C_c^0(X)$ of continuous functions with compact support such that

$$\lim_{k \to \infty} \left\|s_n - g_k^{(n)}\right\|_{L^1(X,\mu)} = 0.$$

Now we choose the diagonal sequence $f_n := g_n^{(n)} \in C_c^0(X)$ for $n \in \mathbb{N}$ and we have that

$$\lim_{n \to \infty} \|f - f_n\|_{L^1(X,\mu)} = 0,$$

i.e. the function f can be approximated by continuous function with compact support. $\square$