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Functional analysis

Solutions to 8. Exercise Sheet

Exercise 1 (Functions of local bounded variation)

For a function $f \in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, $d \in \mathbb{N}$, we define the functional

$$\Phi_f \colon C^1_c\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathbb{R},\ g \mapsto -\int_{\mathbb{R}^d} f(x)\,\text{div}(g)(x)\text{d}x.$$

We call the function f a function of local bounded variation if and only if for every compact subset $K \subseteq \mathbb{R}^d$ it holds

$$C(K) = \sup\left\{\Phi_f(g) \colon g \in C^1_c\left(\mathbb{R}^d, \mathbb{R}^d\right) \text{ with } \text{supp}\,(g) \subseteq K \text{ and } |g(x)| \leq 1 \text{ on } \mathbb{R}^d\right\} < \infty.$$

Show that there is a Radon measure μ_f and a μ_f-measurable function $\eta_f \colon \mathbb{R}^d \to \mathbb{R}^d$ with $|\eta_f| = 1$ μ_f-almost everywhere on $\mathbb{R}^d$ such that we have

$$\Phi_f(g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x)\text{d}\mu_f(x) \text{ for all } g \in C^0_c\left(\mathbb{R}^d, \mathbb{R}^d\right).$$

Prove the identities

$$\mu_f(A) = \mathcal{L}^d \llcorner |\nabla f|\,(A) := \int_A |\nabla f(x)|\,\text{d}x \text{ for all } \mu_f\text{-measurable subsets } A \subseteq \mathbb{R}^d \text{ and } \eta_f = \frac{\nabla f}{|\nabla f|} \text{ for } f \in C^1\left(\mathbb{R}^d\right),$$

Solution of Exercise 1

We denote by $BV_{\text{loc}}\left(\mathbb{R}^d, \mathbb{R}\right)$ the set of $L^1_{\text{loc}}\left(\mathbb{R}^d, \mathbb{R}\right)$ functions of local bounded variation. Let $f \in BV_{\text{loc}}\left(\mathbb{R}^d, \mathbb{R}\right)$ be a function of local bounded variation. The functional Φ_f is linear on $C^1_c\left(\mathbb{R}^d, \mathbb{R}^d\right)$, since the divergenz div and the integral are linear. If $V \subseteq \mathbb{R}^d$ is an open and bounded subset of $\mathbb{R}^d$, then by the theorem of Heine-Borel we know that the subset $\overline{V} \subseteq \mathbb{R}^d$ is compact. Then we know that

$$|\Phi_f(g)| \leq C\left(\overline{V}\right) \|g\|_{L^\infty(V, \mathbb{R}^d)} \text{ for all } g \in C^1_c\left(V, \mathbb{R}^d\right),$$

i.e. Φ_f is a continuous/ bounded linear functional on $C^1_c\left(V, \mathbb{R}^d\right)$ with $\|\Phi_f\| = C\left(\overline{V}\right)$. Now let $K \subseteq \mathbb{R}^d$ be a compact subset of $\mathbb{R}^d$ and choose some open and bounded subset $V \subseteq \mathbb{R}^d$ with $K \subseteq V$. Let $g \in C^0_c\left(\mathbb{R}^d, \mathbb{R}^d\right)$ be arbitrary with $\text{supp}(g) \subseteq K$. Then by Analysis III there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq C^1_c\left(V, \mathbb{R}^d\right)$ such that $g_n \to g$ uniformly on V for $n \to \infty$, i.e.

$$\lim_{n\to\infty} \|g - g_n\|_{L^\infty(V, \mathbb{R}^d)} = 0.$$

By linearity of Φ_f and the estimate above we get that

$$|\Phi_f(g_n) - \Phi_f(g_m)| = |\Phi_f(g_n - g_m)| \leq C\left(\overline{V}\right) \|g_n - g_m\|_{L^\infty(V, \mathbb{R}^d)} \to 0 \text{ for } n, m \to \infty,$$

since $(g_n)_{n\in\mathbb{N}}$ is also a Cauchy-sequence in $L^\infty\left(V, \mathbb{R}^d\right)$. This means that $(\Phi_f(g_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy-sequence in $\mathbb{R}$ and $(\mathbb{R}, |\cdot|)$ is complete, i.e.

$$A := \lim_{n\to\infty} \Phi_f(g_n) \in \mathbb{R} \text{ exists in } \mathbb{R}.$$

If $(h_n)_{n\in\mathbb{N}} \subseteq C^1_c\left(V, \mathbb{R}^d\right)$ is another sequence such that

$$\lim_{n\to\infty} \|g - h_n\|_{L^\infty(V, \mathbb{R}^d)} = 0,$$

then we also get that

$$B := \lim_{n\to\infty} \Phi_f(g_n) \in \mathbb{R} \text{ exists in } \mathbb{R}.$$

We get by the linearity and boundedness of Φ_f and by the convergence of $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ that

$$|A - B| = \lim_{n\to\infty} |\Phi_f(g_n) - \Phi_f(h_n)| = \lim_{n\to\infty} |\Phi_f(g_n - h_n)|$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} C\left(\overline{V}\right)\|g_n - h_n\|_{L^\infty(V,\mathbb{R}^d)}\\
&\leq \lim_{n\to\infty} C\left(\overline{V}\right)\|g_n - g\|_{L^\infty(V,\mathbb{R}^d)} + \lim_{n\to\infty} C\left(\overline{V}\right)\|g - h_n\|_{L^\infty(V,\mathbb{R}^d)}\\
&= 0 + 0 = 0,
\end{aligned}
$$

i.e. $|A - B| = 0$, i.e. $A = B$. This means that the limit of $\Phi_f(g_n)$ for $n \to \infty$ does not depend on the choice of the sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_c^1\left(V, \mathbb{R}^d\right)$. And because of that we can extend our functional Φ_f uniquely to a linear functional (since Φ_f and the limit is also linear)

$$L_f\colon C_c^0\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathbb{R}$$

with

$$\sup\left\{L_f(g)\colon g \in C_c^0\left(V, \mathbb{R}^d\right) \text{ with } \operatorname{supp}(g) \subseteq K \text{ and } |g| \leq 1 \text{ on } V\right\} \leq C\left(\overline{V}\right) < \infty$$

for each compact subset $K \subseteq \mathbb{R}^d$ of $\mathbb{R}^d$ and for each open and bounded subset $V \subseteq \mathbb{R}^d$ with $K \subseteq V$. The Riesz Representation Theorem gives us now a Radon measure μ_f and a μ_f-measurable function $\eta_f\colon \mathbb{R}^d \to \mathbb{R}^d$ with $|\eta_f(x)| = 1$ for μ_f-almost every $x \in \mathbb{R}^d$, such that

$$L_f(g) = \int_{\mathbb{R}^d} g(x)\cdot\eta_f(x)\mathrm{d}\mu_f(x) \text{ for all } g \in C_c^0\left(\mathbb{R}^d, \mathbb{R}^d\right).$$

If addionally the function f is continuously differentiable on $\mathbb{R}^d$, i.e. $f \in C^1\left(\mathbb{R}^d, \mathbb{R}^d\right)$, then we get with Green's formula

$$
\begin{aligned}
\Phi_f(g) &= -\int_{\mathbb{R}^d} f(x)\operatorname{div}(g)(x)\mathrm{d}x = \int_{\mathbb{R}^d} \nabla f(x)\cdot g(x)\mathrm{d}x = \int_{\mathbb{R}^d} g(x)\cdot\frac{\nabla f}{|\nabla f|}(x)\cdot\nabla f(x)\mathrm{d}x\\
&= \int_{\mathbb{R}^d} g(x)\cdot\frac{\nabla f}{|\nabla f|}(x)\mathrm{d}\mathcal{L}^d\llcorner|\nabla f|(x)
\end{aligned}
$$

for all $g \in C_c^1\left(\mathbb{R}^d, \mathbb{R}^d\right)$. If we now extend this to $C_c^0\left(\mathbb{R}^d, \mathbb{R}^d\right)$ we get that

$$\int_{\mathbb{R}^d} g(x)\cdot\eta_f(x)\mathrm{d}\mu_f(x) = L(g) = \int_{\mathbb{R}^d} g(x)\cdot\frac{\nabla f}{|\nabla f|}(x)\mathrm{d}\mathcal{L}^d\llcorner|\nabla f|(x)$$

for $g \in C_c^0\left(\mathbb{R}^d, \mathbb{R}^d\right)$, i.e. $\mu_f = \mathcal{L}^d\llcorner|\nabla f|$ and $\eta_f = \frac{\nabla f}{|\nabla f|}$ μ_f-almost everywhere on $\mathbb{R}^d$. □

Exercise 2 ((C) Weak* convergence)

Let the two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f = \sum_{k=-\infty}^{\infty} \chi_{\left[k, k+\frac{1}{2}\right]},\; g = \sum_{k=-\infty}^{\infty} \chi_{\left[k-\frac{1}{2}, k\right]} \text{ on } \mathbb{R}.$$

Show that the sequences $f_n, g_n, h_n\colon (0,1) \to \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$f_n(x) = f(nx),\; g_n(x) = g(nx) \text{ and } h_n(x) = f_n(x)g_n(x) \text{ for } x \in (0,1)$$

converge weakly* in $L^\infty((0,1))$ to functions f, g resp. a function $h\colon (0,1) \to \mathbb{R}$, but it holds that $h \neq fg$ on $(0,1)$.

Solution of Exercise 2

First we will show a much more general result:
Theorem: Let $I \subseteq \mathbb{R}$ be an open and bounded interval on $\mathbb{R}$ and let $1 < p \leq \infty$ and $1 \leq q < \infty$ be the conjugate Hölder exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. If $\varphi \in L^\infty(\mathbb{R})$ is a periodic function with period $\lambda > 0$, i.e. $\varphi(x+\lambda) = \varphi(x)$ for almost every $x \in \mathbb{R}$, then the function sequence $\varphi_n(x) := \varphi(nx)$, $n \in \mathbb{N}$, $x \in I$, converge weakly* in $L^p(I)'$ to $\frac{1}{\lambda}\int_0^\lambda \varphi(x)\mathrm{d}x$ as $n \to \infty$.
Proof: Set $c := \frac{1}{\lambda}\int_0^\lambda \varphi(x)\mathrm{d}x$. Since $\varphi \in L^\infty(\mathbb{R})$ we get that the function

$$h\colon \mathbb{R} \to \mathbb{R},\; x \mapsto \int_0^x (\varphi(\tau) - c)\,\mathrm{d}\tau$$

is (Lipschitz-)continuous on $\mathbb{R}$, because we have

$$|h(x) - h(y)| = \left|\int_0^x (\varphi(\tau) - c)\,\mathrm{d}\tau - \int_0^y (\varphi(\tau) - c)\,\mathrm{d}\tau\right| = \left|\int_y^x \varphi(\tau)\mathrm{d}\tau + c(y - x)\right|$$

$$
\begin{aligned}
&\le \int_{\min\{x,y\}}^{\max\{x,y\}} |\varphi(\tau)|\,\mathrm{d}\tau + |c|\,|x-y| \le \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\min\{x,y\}}^{\max\{x,y\}} 1\mathrm{d}\tau + |c|\,|x-y| \\
&= \|\varphi\|_{L^\infty(\mathbb{R})} \left|\max\{x,y\} - \min\{x,y\}\right| + |c|\,|x-y| \\
&= \|\varphi\|_{L^\infty(\mathbb{R})} |x-y| + |c|\,|x-y| = \left(\|\varphi\|_{L^\infty(\mathbb{R})} + |c|\right)|x-y|\,.
\end{aligned}
$$

The function h is also bounded on $\mathbb{R}$, since by continuity it is clear that h is bounded on every compact interval on $\mathbb{R}$ and we get for all $k \in \mathbb{N}_0$:

$$
\begin{aligned}
h(k\lambda) &= \int_0^{k\lambda} (\varphi(\tau) - c)\,\mathrm{d}\tau = \int_0^{k\lambda} \varphi(\tau)\mathrm{d}\tau - k\lambda c = \sum_{l=1}^{k} \left[\int_{(l-1)\lambda}^{l\lambda} \varphi(\tau)\mathrm{d}\tau - \lambda c\right] \\
&= \sum_{l=1}^{k} \left[\int_0^{\lambda} \varphi(\tau + (l-1)\lambda)\mathrm{d}\tau - \lambda c\right] \\
&= \sum_{l=1}^{k} \left[\int_0^{\lambda} \varphi(\tau)\mathrm{d}\tau - \lambda c\right] \\
&= \sum_{l=1}^{k} 0 = 0,
\end{aligned}
$$

and $h(-k\lambda) = 0$ by the same calculation, i.e. $L := \sup_{x\in\mathbb{R}} |h(x)| = \sup_{x\in[0,\lambda]} |h(x)| < \infty$, i.e. h is bounded on $\mathbb{R}$. If $a, b \in \mathbb{R}$ and $J := (a,b) \subseteq I$ or $J := [a,b] \subseteq I$ is an arbitrary interval we get that

$$
\begin{aligned}
\left|\int_I (\varphi_n(x) - c)\,\chi_J(x)\mathrm{d}x\right| &= \left|\int_a^b (\varphi(nx) - c)\,\mathrm{d}x\right| = \frac{1}{n}\left|\int_{na}^{nb} (\varphi(x) - c)\,\mathrm{d}x\right| \\
&= \frac{1}{n}\,|h(nb) - h(na)| \le \frac{2L}{n} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

This implies now that if $\psi \colon \mathbb{R} \to \mathbb{R}$ is a stepfunction we have that

$$
\lim_{n\to\infty} \int_I (\varphi_n(x) - c)\,\psi(x)\mathrm{d}x = 0
$$

and we know that

$$
\|\varphi_n\|_{L^\infty(I)} = \|\varphi(n\cdot)\|_{L^\infty(I)} \le \|\varphi\|_{L^\infty(\mathbb{R})} < \infty
$$

for all $n \in \mathbb{N}$, i.e. $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{L^\infty(I)} \le \|\varphi\|_{L^\infty(I)} < \infty$. Then we get by Exercise 4 (2') that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ converge weakly to c in $L^p(I)$ as $n \to \infty$, i.e. $(\varphi_n)_{n\in\mathbb{N}}$ converge weakly* to c in $L^p(I)'$ as $n \to \infty$. □

The functions f and g are obviously bounded by 1 and are periodic with period $\lambda = 1 > 0$, since for $x \in \mathbb{R}$ we have two cases: Case 01.: $f(x) = 1$ resp. $g(x) = 1$.

Then there is some $k_0 \in \mathbb{Z}$ resp. $l_0 \in \mathbb{Z}$ with $x \in \left[k_0, k_0 + \frac{1}{2}\right]$ resp. $x \in \left[l_0 - \frac{1}{2}, l_0\right]$, and we get that $x + 1 \in \left[k_0 + 1, k_0 + 1 + \frac{1}{2}\right]$ resp. $x + 1 \in \left[l_0 + 1 - \frac{1}{2}, l_0 + 1\right]$, i.e.

$$
f(x+1) = 1 = f(x),\ \text{resp. } g(x+1) = 1 = g(x).
$$

Case 02.: $f(x) = 0$ resp. $g(x) = 0$.

Then for all $k, l \in \mathbb{Z}$ we have that $x \notin \left[k, k + \frac{1}{2}\right]$ resp. $x \notin \left[l - \frac{1}{2}, l\right]$, and we get that $x + 1 \notin \left[k + 1, k + 1 + \frac{1}{2}\right]$ resp. $x + 1 \notin \left[l + 1 - \frac{1}{2}, l + 1\right]$ for all $k, l \in \mathbb{Z}$, i.e.

$$
f(x+1) = 0 = f(x),\ \text{resp. } g(x+1) = 0 = g(x).
$$

And we calculate

$$
\begin{aligned}
\frac{1}{1}\int_0^1 f(x)\mathrm{d}x &= \int_0^1 \sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{2}]}(x)\mathrm{d}x = \int_0^1 \chi_{[0,\frac{1}{2}]}(x)\mathrm{d}x = \int_0^{\frac{1}{2}} 1\mathrm{d}x = \frac{1}{2} - 0 = \frac{1}{2}, \\
\frac{1}{1}\int_0^1 g(x)\mathrm{d}x &= \int_0^1 \sum_{k=-\infty}^{\infty} \chi_{[k-\frac{1}{2},k]}(x)\mathrm{d}x = \int_0^1 \chi_{[1-\frac{1}{2},1]}(x)\mathrm{d}x = \int_0^1 \chi_{[\frac{1}{2},1]}(x)\mathrm{d}x = \int_{\frac{1}{2}}^1 1\mathrm{d}x = 1 - \frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

By the previous result we know that $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge weakly* to $\frac{1}{2}$ in $L^\infty((0,1))$ for $n \to \infty$. We define the function $h \colon \mathbb{R} \to \mathbb{R}$ by $h = fg$ on $\mathbb{R}$, then it is $h_n = h(n\cdot)$. The function h is obviously bounded by 1 and is periodic with periode $\lambda = 1 > 0$, since we have $h(x+1) = f(x+1)g(x+1) = f(x)g(x) = h(x)$ for all $x \in \mathbb{R}$ by the periodicity of f and g. For the integral we get

$$\frac{1}{1}\int_0^1 h(x)\mathrm{d}x = \int_0^1 f(x)g(x)\mathrm{d}x = \int_0^1 \left(\sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{2}]}(x)\right) \cdot \left(\sum_{k=-\infty}^{\infty} \chi_{[k-\frac{1}{2},k]}(x)\right) \mathrm{d}x$$

$$= \int_0^1 \chi_{[0,\frac{1}{2}]}(x) \cdot \chi_{[1-\frac{1}{2},1]}(x)\mathrm{d}x = \int_0^1 \chi_{[0,\frac{1}{2}]}(x) \cdot \chi_{[\frac{1}{2},1]}(x)\mathrm{d}x = \int_0^1 \chi_{\{\frac{1}{2}\}}(x)\mathrm{d}x = 0,$$

since $\left\{\frac{1}{2}\right\}$ has measure zero. By the previous result we get that the sequence $(h_n)_{n\in\mathbb{N}}$ converge weakly* to 0 in $L^\infty((0,1))$, but $0 \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$. □

Exercise 3 (Weak convergence)

For every $\varepsilon > 0$ we define

$$f_\varepsilon \colon \mathbb{R} \to \mathbb{R},\ \ f_\varepsilon(x) = \sqrt{\frac{\varepsilon}{x^2+\varepsilon^2}}.$$

Show that for every $\varepsilon > 0$ we have that $\|f_\varepsilon\|_{L^2(\mathbb{R})} = \sqrt{\pi}$. Check the weak convergence of f_ε and f_ε^2 in $L^2(\mathbb{R})$ for $\varepsilon \to 0^+$.

Exercise 4 ((C) Weak convergence in $L^2((-\pi,\pi))$)

(1) Show that if $(u_n)_{n\in\mathbb{N}} \subseteq L^2((-\pi,\pi))$ is a bounded sequence such that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} u_n(x)\varphi(x)\mathrm{d}x = 0 \text{ for all } \varphi \in C_c^\infty((-\pi,\pi)),$$

the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2((-\pi,\pi))$ if $n \to \infty$.

(2) Check if the sequence $u_n(x) := \sin(nx),\ n \in \mathbb{N},\ x \in (-\pi,\pi)$, converges pointwise on $(-\pi,\pi)$ and/ or converges weakly in $L^2((-\pi,\pi))$ if $n \to \infty$.