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Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Karlsruhe Institute of Technology

Functional analysis

10. Exercise Sheet

Exercise 1 (Sobolev spaces are reflexive and seperable)

Show that the Sobolev space $W^{1,p}(\Omega)$ is seperable for $1 \leqslant p < \infty$ and is reflexive for $p \in (1,\infty)$.

(Hint: Use the map $J\colon W^{1,p}(\Omega) \to X_p(\Omega)\,,\ u \mapsto (u, \nabla u)$, where $X_p(\Omega) := \bigtimes_{i=1}^{d+1} L^p(\Omega)$.)

Exercise 2 ((C) Sobolev spaces in Hölder spaces)

Show that for the real unit interval $I = (0,1) \subseteq \mathbb{R}$ the Sobolev space $W^{1,p}(I)$, $p \in [1,\infty]$, is continuously embedded in the Hölder space $C^{0,1-\frac{1}{p}}(I)$.

Exercise 3 ($D^2u = 0$ implies affin linearity)

Let $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^d)$ be a Sobolev function with $D^2u = 0$ on $\mathbb{R}^d$. Show that the function u is affin linear, i.e. there are two constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ such that

$$u(x) = a + \langle b, x\rangle \text{ for all } x \in \mathbb{R}^d.$$

Exercise 4 ((C) Chain rule for weak derivatives)

Let $u \in L^1_{\text{loc}}(\Omega)$ be a function with $Du \in L^1_{\text{loc}}(\Omega)$. Show that for the absolute value of u we have $|u| \in W^{1,1}_{\text{loc}}(\Omega)$ with

$$D(|u|) = \operatorname{sign}(u)\, Du \text{ on } \Omega.$$

(Hint: Approximate the absolute value function by $f_\varepsilon(z) = \sqrt{z^2+\varepsilon^2}$, $z \in \mathbb{R}$, for $\varepsilon > 0$.)