12.01.2017
Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Karlsruhe Institute of Technology

Functional analysis

11. Exercise Sheet

Exercise 1 ((C) Projection into the quotient space)

Let $(X, \|\cdot\|_X)$ be a Banach space and $V \subseteq X$ be a closed subspace of X. We equip the quotient space X/V with the norm

$$\|[x]\|_{X/V} := \inf_{v\in V} \|x+v\|_X ,$$

then $\left(X/V, \|\cdot\|_{X/V}\right)$ is a Banach space. Show that the projection $P\colon X \to X/V,\ x \mapsto [x]$ is an open map with norm $\|P\| \leq 1$. Conclude that $\|P\| = 1$ if and only if $V \neq X$, otherwise it is zero.

Exercise 2 (Weak formulation of the Neumann problem)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary and outer unit normal ν, the coefficients $a_{ij} \in C^1\left(\overline{\Omega}\right),\ b_i, c, f \in C^0\left(\overline{\Omega}\right)$. Show that $u \in C^2\left(\overline{\Omega}\right)$ solves the Neumann-problem

$$\begin{cases} -\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij}\partial_j u\right) + \sum_{i=1}^d b_i \partial_i u + cu = f & \text{in } \Omega \\ \sum_{i=1}^d \nu_i \sum_{j=1}^d a_{ij}\partial_j u = 0 & \text{on } \partial\Omega \end{cases},$$

if and only if for all $\varphi \in C^\infty\left(\overline{\Omega}\right)$ it holds

$$\int_\Omega \left[\sum_{i=1}^d \partial_i\varphi(x) \sum_{j=1}^d a_{ij}(x)\partial_j u(x) + \varphi(x)\left(\sum_{i=1}^d b_i(x)\partial_i u(x) + c(x)u(x)\right)\right] \mathrm{d}x = \int_\Omega \varphi(x) f(x) \mathrm{d}x.$$

Exercise 3 (Dual space of $W_0^{1,2}(\Omega)$)

Let $X = L^2(\Omega) \times L^2\left(\Omega, \mathbb{R}^d\right)$ be the product space and we choose the norms $\|\cdot\|_X$, $\|\cdot\|_{W^{1,2}(\Omega)}$ such that the embedding $J\colon W_0^{1,2}(\Omega) \to X,\ u \mapsto (u, \nabla u)$ is isometric. Define the map

$$P\colon X \to W_0^{1,2}(\Omega)',\ P(f,F)(u) = \int_\Omega \left(f(x)u(x) + \langle F(x), \nabla u(x)\rangle\right).$$

Show that:

(1) Orthogonal decomposition: $X = \ker(P) \oplus \mathrm{range}(J)$.

(2) The map P is surjective and for $\varphi \in W_0^{1,2}(\Omega)'$ we have

$$\|\varphi\| = \inf\left\{\|(f,F)\|_X : P(f,F) = \varphi\right\}.$$

Exercise 4 ((C) Green Operator)

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded and $a \in L^\infty\left(\Omega, M_d(\mathbb{R})\right)$ be elliptic. Consider the operator

$$G = I \circ L^{-1} \circ I'\colon L^2(\Omega) \to L^2(\Omega),$$

where

$$
\begin{aligned}
&L\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)',\ (Lv)(u) = \int_\Omega \langle \nabla u(x), a(x)\nabla v(x)\rangle \, \mathrm{d}x, \\
&I\colon W_0^{1,2}(\Omega) \to L^2(\Omega),\ Iv = v, \\
&I'\colon L^2(\Omega) \to W_0^{1,2}(\Omega)',\ (I'f)(u) = \int_\Omega f(x)u(x)\mathrm{d}x.
\end{aligned}
$$

Show that $LGf = f$ for $f \in L^2(\Omega)$ and show that if a is symmetric, then the operator G is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, i.e. $G = G^*$.