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Functional analysis

Solutions to 12. Exercise Sheet

Definition: (Subspace of compact operators) Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two normed spaces. We set the subset

$$K(X,Y) := \{K \in L(X,Y) \colon K \text{ is a compact operator}\} \subseteq L(X,Y)$$

of compact operators.

Lemma: Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces, then $K(X,Y)$ is a closed linear subspace of $L(X,Y)$.

Remark: If $K \subseteq L(X,Y)$ has finite-dimensional range, i.e. $\dim(\text{range}(K)) < \infty$, the operator K is compact.

Exercise 1 ((C) Hilbert-Schmidt-Integraloperators)

Let $I := (0,1) \subseteq \mathbb{R}$ be the open unit interval and $k \in L^2(I \times I, \mathbb{C})$ be the kernel of the operator

$$K\colon L^2(I,\mathbb{C}) \to L^2(I,\mathbb{C}),\ (Kf)(x) = \int_I k(x,y)f(y)\mathrm{d}y.$$

Show that:

(1) The operator K is well-defined and bounded with $\|K\| \le \|k\|_{L^2(I\times I,\mathbb{C})}$.

(2) The adjoint operator K^* of K has the integralrepresentation

$$(K^*f)(x) = \int_I k^*(x,y)f(y)\mathrm{d}y \text{ for } f \in L^2(I,\mathbb{C}),\ x \in I,$$

where $k^*(x,y) := \overline{k(y,x)}$ for all $(x,y) \in I \times I$.

(3) The operator K is compact.
(Hint: Use the fact that the set of $(x,y) \mapsto \lambda\chi_{[a,b]}(x)\chi_{[\alpha,\beta]}(y)$, $\lambda \in \mathbb{C}$, is dense in $L^2(I \times I, \mathbb{C})$.)

Solution of Exercise 1

(1) The operator K is linear, since we have for $f, g \in L^2(I,\mathbb{C})$ and $\lambda, \mu \in \mathbb{C}$:

$$\begin{aligned}(K(\lambda f + \mu g))(x) &= \int_I k(x,y)(\lambda f(y) + \mu g(y))\,\mathrm{d}y \\ &= \lambda \int_I k(x,y)f(y)\mathrm{d}y + \mu \int_I k(x,y)f(y)\mathrm{d}y = \lambda(Kf)(x) + \mu(Kg)(x) \\ &= (\lambda Kf + \mu Kg)(x)\end{aligned}$$

for all $x \in I$.
The operator K is well-defined, since we have for all $f \in L^2(I,\mathbb{C})$ by the Hölder inequality ($p = q = 2$):

$$\begin{aligned}\int_I \left|\int_I k(x,y)f(y)\mathrm{d}y\right|^2 \mathrm{d}x &\le \int_I \left(\int_I |k(x,y)\cdot f(y)|\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &\le \int_I \left(\int_I |k(x,y)|^2\,\mathrm{d}y\right)\cdot\left(\int_I |f(y)|^2\,\mathrm{d}y\right)\mathrm{d}x\end{aligned}$$

$$= \int_I \int_I |k(x,y)|^2 \,\mathrm{d}y\mathrm{d}x \, \|f\|^2_{L^2(I,\mathbb{C})}$$
$$= \int_{I\times I} |k(x,y)|^2 \,\mathrm{d}(x,y) \, \|f\|^2_{L^2(I,\mathbb{C})} = \|k\|^2_{L^2(I\times I,\mathbb{C})} \|f\|^2_{L^2(I,\mathbb{C})} < \infty.$$

This implies directly that

$$\|K\| \leq \|k\|_{L^2(I\times I,\mathbb{C})},$$

i.e. $K \in L\left(L^2(I,\mathbb{C}), L^2(I,\mathbb{C})\right)$. □

(2) For two functions $f, g \in L^2(I,\mathbb{C})$ we have with the theorem of Fubini

$$\begin{aligned}
\langle Kf, g\rangle_{L^2(I,\mathbb{C})} &= \int_I Kf(x)\cdot\overline{g(x)}\mathrm{d}x \\
&= \int_I \left(\int_I k(x,y)f(y)\mathrm{d}y\right)\cdot\overline{g(x)}\mathrm{d}x \\
&= \int_I\int_I k(x,y)f(y)\overline{g(x)}\mathrm{d}y\mathrm{d}x \\
&= \int_I\int_I k(x,y)f(y)\overline{g(x)}\mathrm{d}x\mathrm{d}y \\
&= \int_I f(y)\cdot\left(\int_I k(x,y)\overline{g(x)}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_I f(y)\cdot\overline{\int_I \overline{k(x,y)}g(x)\mathrm{d}x}\mathrm{d}y,
\end{aligned}$$

i.e. the adjoint operator of K is defined by

$$K^*g(x) := \int_I \overline{k(y,x)}g(y)\mathrm{d}y = \int_I k^*(x,y)g(y)\mathrm{d}y \text{ for } g \in L^2(I,\mathbb{C}),$$

where the integral kernel $k^*(x,y) := \overline{k(y,x)}$ for $x, y \in I$. □

(3) Since the subspace $\operatorname{lin}\left\{(x,y)\mapsto \chi_{[a,b]}(x)\chi_{[\alpha,\beta]}(y) \mid a,b,\alpha,\beta\in I\right\}$ is dense in $L^2(I\times I,\mathbb{C})$ we can approximate the kernel k by a sequence $(k_n)_{n\in\mathbb{N}} \subseteq \operatorname{lin}\left\{(x,y)\mapsto \chi_{[a,b]}(x)\chi_{[\alpha,\beta]}(y) \mid a,b,\alpha,\beta\in I\right\}$, i.e.

$$\lim_{n\to\infty}\|k_n - k\|_{L^2(I\times I,\mathbb{C})} = 0,\ k_n(x,y) = \sum_{l=1}^{N_n}\lambda_l^{(n)}\chi_{\left[a_l^{(n)},b_l^{(n)}\right]}(x)\chi_{\left[\alpha_l^{(n)},\beta_l^{(n)}\right]}(y) \text{ for } x,y\in I$$

with $N_n\in\mathbb{N}$, $\left(\lambda_l^{(n)}\right)_{l=1,\ldots,N_n}\subseteq\mathbb{R}$, $a_l^{(n)}, b_l^{(n)}, \alpha_l^{(n)}, \beta_l^{(n)} \in I$ for all $l = 1,\ldots,N_n$ and all $n\in\mathbb{N}$. We define for $n\in\mathbb{N}$ the operators

$$K_n\colon L^2(I,\mathbb{C})\to L^2(I,\mathbb{C}),\ f\mapsto\int_I k_n(\cdot,y)f(y)\mathrm{d}y.$$

Then we have for any $n\in\mathbb{N}$ and $f\in L^2(I,\mathbb{C})$:

$$K_nf = \int_I k_n(\cdot,y)f(y)\mathrm{d}y = \sum_{l=1}^{N_n}\lambda_l^{(n)}\chi_{\left[a_l^{(n)},b_l^{(n)}\right]}(\cdot)\int_{\alpha_l^{(n)}}^{\beta_l^{(n)}} f(y)\mathrm{d}y.$$

The operators K_n, $n\in\mathbb{N}$, are well-defined, since by (1) with kernel k_n:

$$\|K_n\| \leq \|k_n\|_{L^2(I\times I,\mathbb{C})} = \sum_{l=1}^{N_n}\left|\lambda_l^{(n)}\right|\sqrt{\left(b_l^{(n)}-a_l^{(n)}\right)\cdot\left(\beta_l^{(n)}-\alpha_l^{(n)}\right)} < \infty$$

and $\operatorname{range}(K_n) \subseteq \operatorname{lin}\left\{\chi_{\left[a_1^{(n)},b_1^{(n)}\right]},\ldots,\chi_{\left[a_{N_n}^{(n)},b_{N_n}^{(n)}\right]}\right\}$, i.e. $\dim(\operatorname{range}(K_n)) \leq N_n < \infty$, therefore K_n, $n\in\mathbb{N}$, is a compact operator. By Hölder inequality ($p = q = 2$) we get

$$\begin{aligned}
\|K_nf - Kf\|^2_{L^2(I,\mathbb{C})} &= \int_I\left|\int_I k_n(x,y)f(y)\mathrm{d}y - \int_I k(x,y)f(y)\mathrm{d}y\right|^2\mathrm{d}x \\
&= \int_I\left|\int_I (k_n(x,y)-k(x,y))\,f(y)\mathrm{d}y\right|^2\mathrm{d}x \\
&\leq \int_I\left(\int_I |k_n(x,y)-k(x,y)|\,|f(y)|\,\mathrm{d}y\right)^2\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_I \left(\int_I |k_n(x,y) - k(x,y)|^2 \,\mathrm{d}y \right) \cdot \left(\int_I |f(y)|^2 \,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_I \int_I |k_n(x,y) - k(x,y)|^2 \,\mathrm{d}y\mathrm{d}x \, \|f\|^2_{L^2(I,\mathbb{C})} \\
&= \int_{I\times I} |k_n(x,y) - k(x,y)|^2 \,\mathrm{d}(x,y) \, \|f\|^2_{L^2(I,\mathbb{C})} = \|k_n - k\|^2_{L^2(I\times I,\mathbb{C})} \|f\|^2_{L^2(I,\mathbb{C})} \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

i.e.

$$\lim_{n\to\infty} \|K_n - K\| = 0.$$

By the above Lemma we know that $K \in K\left(L^2(I,\mathbb{C}), L^2(I,\mathbb{C})\right)$, this means that K is compact. □

Exercise 2 (The adjoint of the gradient)

What is the adjoint operator of the gradient

$$\nabla \colon W_0^{1,2}(\Omega) \to L^2\left(\Omega, \mathbb{R}^d\right)?$$

Solution of Exercise 2

We consider the operator ∇ as an unbounded operator on $L^2(\Omega)$ with the dense domain $\operatorname{dom}(\nabla) = W_0^{1,2}(\Omega)$. Define the space

$$H^{\operatorname{div}}(\Omega) := \left\{ F \in L^2\left(\Omega, \mathbb{R}^d\right) : \operatorname{div}(F) \in L^2(\Omega) \text{ exists} \right\} \subseteq L^2\left(\Omega, \mathbb{R}^d\right),$$

where we define the (weak) divergenz of an $L^1_{\text{loc}}(\Omega)^d$ function analougsly to the weak derivative (motivated by the Gauß Theorem), i.e. a vector-valued function $F \in L^1_{\text{loc}}(\Omega)^d$ has a weak divergenz if and only if, there is a function $h \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega \langle \nabla\eta(x), F(x) \rangle \,\mathrm{d}x = -\int_\Omega \eta(x) h(x) \mathrm{d}x \text{ for all } \eta \in C_c^\infty(\Omega),$$

in this case $\operatorname{div}(F) := h$. Let $F \in H^{\operatorname{div}}(\Omega)$, then we have for all $f \in C_c^\infty(\Omega)$:

$$\langle f, -\operatorname{div}(F) \rangle_{L^2(\Omega)} = \langle \nabla f, F \rangle_{L^2(\Omega,\mathbb{R}^d)},$$

i.e. $-\operatorname{div} = \nabla|^*_{W_0^{1,2}(\Omega)}$ on $H^{\operatorname{div}}(\Omega)$ and $H^{\operatorname{div}}(\Omega) \subseteq \operatorname{dom}\left(\nabla|^*_{W_0^{1,2}(\Omega)}\right)$.
Now let $F \in \operatorname{dom}\left(\nabla|^*_{W_0^{1,2}(\Omega)}\right) \subseteq L^2\left(\Omega, \mathbb{R}^d\right)$, i.e. there is some function $h \in L^2(\Omega)$ such that

$$\langle \nabla f, F \rangle_{L^2(\Omega,\mathbb{R}^d)} = \langle f, h \rangle_{L^2(\Omega)} = -\langle f, -h \rangle_{L^2(\Omega)} \quad \text{for all } f \in W_0^{1,2}(\Omega),$$

This means that $h = -\operatorname{div}(F)$ and we get that $F \in H^{\operatorname{div}}(\Omega)$, i.e. $\operatorname{dom}\left(\nabla|^*_{W_0^{1,2}(\Omega)}\right) = H^{\operatorname{div}}(\Omega)$ and we get for the adjoint-operator of ∇:

$$\nabla|^*_{W_0^{1,2}(\Omega)} = -\operatorname{div}|_{H^{\operatorname{div}}(\Omega)},$$

where $-\operatorname{div}$ is also an unbounded operator with dense domain. □

Remark (to my notation): I wrote $\nabla|_{W_0^{1,2}(\Omega)}$ instead of ∇ to say what's the domain I consider. Since you can also look at a domain such that $W_0^{1,2}(\Omega) \subseteq D := \operatorname{dom}(\nabla) \subseteq W^{1,2}(\Omega)$. In this case you also get as the adjoint operator $\nabla|^*_D = -\operatorname{div}|_{\widetilde{D}}$, but with domain $H_0^{\operatorname{div}}(\Omega) \subseteq \widetilde{D} := \operatorname{dom}(\nabla|^*_D) \subseteq H^{\operatorname{div}}(\Omega)$. If we choose ∇ with a greater domain, then the domain of ∇^* will be smaller, i.e.

$$D = W^{1,2}(\Omega) \Leftrightarrow \widetilde{D} = H_0^{\operatorname{div}}(\Omega).$$

Exercise 3 (Some Perturbation)

Let $I = (0,\infty) \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ be a constant. Define the operator $A_\lambda \colon W_0^{1,2}(I) \to L^2(I),\ u \mapsto u' + \lambda u$. Show that

(1) $\|u'\|_{L^2(I)} \leq \|A_\lambda u\|_{L^2(I)}$, $\|u\|_{L^2(I)} \leq \frac{1}{|\lambda|} \|A_\lambda u\|_{L^2(I)}$.

(2) The operator A_λ is injective and the image $\operatorname{range}(A_\lambda) \subseteq L^2(I)$ is closed.

(3) It is $(\operatorname{range}(A_\lambda))^\perp = \{0\}$ and $\operatorname{ind}(A_\lambda) = 0$ for $\lambda > 0$, $(\operatorname{range}(A_\lambda))^\perp = \operatorname{lin}\left\{x \mapsto e^{\lambda x}\right\}$ and $\operatorname{ind}(A_\lambda) = -1$ for $\lambda < 0$.

(4) The operator $A_0 u := u'$ (case $\lambda = 0$) is injective with dense image range $(A_0) \subseteq L^2(I)$, but the image range $(A_0) \subseteq L^2(I)$ is not closed.

Solution of Exercise 3

To part (3): My answer was not as good as I want, here is the correct version of why a weak solution of $u' = \lambda u$ implies directly $u(x) = c\mathrm{e}^{\lambda x}$.
Look at the mollified function $\eta_\delta * u$. We have

$$(\eta_\delta * u)' = \eta_\delta * u' = \eta_\delta * (\lambda u) = \lambda \eta_\delta * u.$$

Since $\eta_\delta * u \in C^\infty(I)$ this is an ODE, which we can solve with solution $\eta_\delta * u = c\mathrm{e}^{\lambda \cdot}$ for some constant $c \in \mathbb{R}$. By local convergence we know that

$$c\mathrm{e}^{\lambda \cdot} = \eta_\delta * u \to u \text{ in } W^{1,2}_{0,\mathrm{loc}}(I) \text{ as } \delta \to 0^+,$$

i.e. $u = c\mathrm{e}^{\lambda \cdot}$ almost everywhere on I.
Therefore my argumentation was correct in the proof. $\square$

Exercise 4 ((C) To the Ehrling-Lemma)

1. Let $R > 0$ be a radius and $\Omega := B_R(0) \subseteq \mathbb{R}^d$. Show that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $u \in C^2\left(\overline{\Omega}\right)$ we have
$$\|\nabla u\|_{C^0(\Omega)} \leq \varepsilon \left\|D^2 u\right\|_{C^0(\Omega)} + C_\varepsilon \|u\|_{C^0(\Omega)}$$
and give a concrete estimate for the constant C_ε.

2. Let $u \in C^2([0,1])$ be a solution of the linear differential equation
$$a u'' + b u' + d u = 0 \text{ in } (0,1)$$
with coefficients $a, b, d \in C^0([0,1])$ and $a \geq c_0 > 0$ on $[0,1]$, where $c_0 > 0$ is a positive constant. Then there is a constant $C = C(a,b,d) > 0$ such that
$$\|u\|_{C^2([0,1])} \leq C \|u\|_{C^0([0,1])}.$$