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Functional analysis

Solutions to 13. Exercise Sheet

Exercise 1 ((C) Infinite linear system)

Show that the infinite linear system

$$x_i + \sum_{j=1}^{\infty} a_{ij} x_j = b_i \text{ for } i \in \mathbb{N}$$

has for every $b \in l^2(\mathbb{N}, \mathbb{R})$ an unique solution $x \in l^2(\mathbb{N}, \mathbb{R})$, if the matrix $(a_{ij})_{1 \le i,j \le N}$ is for every $N \in \mathbb{N}$ positive semi-definite with $\sum\limits_{i,j=1}^{\infty} a_{ij}^2 < \infty$.

Exercise 2 (Solvability of the Neumann-Problem)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary and outer unit normal ν, $f \in L^2(\Omega)$. For functions $a \in L^\infty(\Omega, M_d(\mathbb{R}))$ and $q \in L^\infty(\Omega)$ we define the operator $L \colon W^{1,2}(\Omega) \to W^{1,2}(\Omega)'$ by

$$(Lv)(u) = \int_\Omega \left[\langle a(x)\nabla v(x), \nabla u(x)\rangle + q(x)v(x)u(x)\right] \mathrm{d}x \text{ for } u, v \in W^{1,2}(\Omega).$$

Additionally $a(\cdot)$ is symmetric and elliptic on Ω with some constant $\mu > 0$. Show that:

(1) The operator L is an isomorphism, if $q(x) \ge \lambda > 0$ for all $x \in \Omega$ and some $\lambda > 0$.

(2) The operator L is a Fredholm-Operator with index zero.
(Hint: Use without proof that the embedding $W^{1,2}(\Omega) \subseteq L^2(\Omega)$ is compact, if $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with C^1-boundary)

(3) The condition $\int_\Omega f(x)\mathrm{d}x = 0$ is necessary and sufficient for the existence of a weak solution to the Neumann-Problem

$$\begin{cases} -\operatorname{div}(a\nabla v) = f & \text{in } \Omega \\ \sum\limits_{i=1}^{d} \nu_i \sum\limits_{j=1}^{d} a_{ij}\partial_j v = 0 & \text{on } \partial\Omega \end{cases}$$

Solution of Exercise 2

Question: Why does the solution solves the boundary condition?
We get for all $\eta \in W^{1,2}(\Omega)$:

$$\int_\Omega f(x)\eta(x)\mathrm{d}x = \int_\Omega \langle a(x)\nabla u(x), \nabla\eta(x)\rangle \,\mathrm{d}x,$$

i.e. we the weak divergenz of $a\nabla u$ is equal to the L^2-function $-f$ on Ω. By the weak formulation of the Gauß theorem (see the Remark below) we know for all $\eta \in C^\infty\left(\overline{\Omega}\right)$:

$$\begin{aligned} 0 &= \int_\Omega -\operatorname{div}(a\nabla u)(x)\eta(x)\mathrm{d}x - \int_\Omega f(x)\eta(x)\mathrm{d}x \\ &= \int_\Omega \langle a(x)\nabla u(x), \nabla\eta(x)\rangle \,\mathrm{d}x - \int_\Omega f(x)\eta(x)\mathrm{d}x + \int_{\partial\Omega} \langle \nu(x), a(x)\nabla u(x)\rangle\, \eta(x)\mathrm{d}s(x) \\ &= \int_{\partial\Omega} \langle \nu(x), a(x)\nabla u(x)\rangle\, \eta(x)\mathrm{d}s(x), \end{aligned}$$

i.e. by the fundamental lemma of calculus:

$$\langle \nu(x), a(x)\nabla u(x)\rangle = 0 \text{ for almost all } x \in \partial\Omega.$$

□

Exercise 3 (Solvability criteria)

Consider the operator $L\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)'$, $L = L_0 + K$ defined in the lecture with measurable coefficients a, b, c, q and let the operator L_0 be elliptic with constant $\mu > 0$. Show for every $\varphi \in W_0^{1,2}(\Omega)'$ the equivalence of the following statements (Theorem 10.20 in the lecture):

(1) $Lv = \varphi$ has a solution $v \in W_0^{1,2}(\Omega)$.

(2) $\varphi(u) = 0$ for all $u \in \ker(L^*)$.

Here $L^* := L' \circ J$, where $J\colon W^{1,2}(\Omega) \to W^{1,2}(\Omega)''$ is the canonical embedding. What is the meaning of (2), if the right-hand side is some L^2-function f?

Exercise 4 ((C) The spectrum of a multiplication operator)

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$, $X := C_b^0(\Omega) := \{f\colon \Omega \to \mathbb{K} |\ f \text{ is continuous and bounded on } \Omega\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in X$ and T_m be the multiplication operator on X, i.e.

$$T_m\colon X \to X,\ f \mapsto m \cdot f.$$

What is the spectrum $\sigma(T_m)$? Determine the type of the spectral values and the resolvent $R(\lambda, T_m)$ for all $\lambda \in \rho(T_m)$.

Solution of Exercise 4

Claim 1.: It is $\sigma(T_m) = \overline{m(\Omega)}$.
Proof: Let $\lambda \notin \overline{m(\Omega)}$, then the function $\frac{1}{\lambda - m} \in X$, and the multiplication operator

$$R_\lambda f := \frac{1}{\lambda - m} f$$

is in $L(X, X)$ and satisfies

$$(\lambda \mathrm{Id}_X - T_m) R_\lambda = R_\lambda (\lambda \mathrm{Id}_X - T_m) = \mathrm{Id}_X,$$

i.e. $\lambda \in \rho(T_m)$ with resolvent

$$R(\lambda, T_m) f = R_\lambda f = \frac{1}{\lambda - m} f \text{ for } f \in X.$$

This implies now that $\sigma(T_m) \subseteq \overline{m(\Omega)}$.
Let $\lambda \in m(\Omega)$. Then there is some $x_0 \in \Omega$ such that $\lambda = m(x_0)$. Then we get for all $f \in X$:

$$[(\lambda \mathrm{Id}_X - T_m) f](x_0) = \lambda f(x_0) - m(x_0) f(x_0) = 0.$$

This means that $\lambda \mathrm{Id}_X - T_m$ cannot be surjective on X, since otherwise every function in X would be zero in x_0, i.e. $\lambda \in \sigma(T_m)$. This implies now:

$$\overline{m(\Omega)} \subseteq \overline{\sigma(T_m)} = \sigma(T_m),$$

since the spectrum is closed (see lecture).
All together we get: $\sigma(T_m) = \overline{m(\Omega)}$. □
Claim 2.: $\sigma_c(T_m) = \emptyset$, T_m has eigenvalues if and only if the function m has locally constant parts; $\sigma(T_m) = \sigma_p(T_m) \dot{\cup} \sigma_{res}(T_m)$.
Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of T_m. Then there is a function $f \in X \backslash \{0\}$ such that $T_m f = \lambda f$. This implies now:

$$(\lambda - m) f = 0 \text{ on } \Omega.$$

So there has to be an open neighborhood $U \subseteq \mathbb{R}^d$ of some point $x_0 \in \Omega$ such that f is not zero on $\Omega \cap U$. This gives us that $\lambda = m$ on $\Omega \cap U$, i.e. m has a part, where it is locally constant. If otherwise there is a radius $R > 0$, point $x_0 \in \Omega$ such that m is constant to some $\lambda \in \mathbb{C}$ on $\Omega \cap B_R(x_0)$, then we can find a function such that $T_m f = \lambda f$, since there is a continuous function f such that f is constant to 1 on $\Omega \cap B_{\frac{R}{2}}(x_0)$ and constant to zero on $\Omega \backslash B_R(x_0)$.
If there would be some $\lambda \in \sigma_c(T_m)$, then the operator $\lambda \mathrm{Id}_X - T_m$ would have dense range. Let $(x_n)_{n \in \mathbb{N}} \subseteq \Omega$ be a sequence such that $\lim_{n \to \infty} m(x_n) = \lambda$. But, since functions in X are bounded, we now that

$$|(\lambda - m(x_n)) f(x_n)| \leq \|f\|_{C^0(\Omega)} |\lambda - m(x_n)| \to 0 \text{ as } n \to \infty \text{ for all } f \in X.$$

This implies now

$$\sup_{x \in \Omega} |1 - (\lambda - m(x)) f(x)| \geq \frac{1}{2} > 0 \text{ for all } f \in X,$$

and $1 \in X$, i.e. $\operatorname{range}(\lambda \mathrm{Id}_X - T_m)$ cannot be dense in X. Therefore $\sigma_c(T_m) = \emptyset$. $\square$

Definition: (The space H^{div}) Define the space

$$H^{\mathrm{div}}(\Omega) := \left\{F \in L^2\left(\Omega, \mathbb{R}^d\right) : \operatorname{div}(F) \in L^2(\Omega) \text{ exists}\right\} \subseteq L^2\left(\Omega, \mathbb{R}^d\right),$$

where we define the (weak) divergenz of an $L^1_{\mathrm{loc}}(\Omega)^d$ function analougsly to the weak derivative (motivated by the Gauß Theorem), i.e. a vector-valued function $F \in L^1_{\mathrm{loc}}(\Omega)^d$ has a weak divergenz if and only if, there is a function $h \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \langle \nabla\eta(x), F(x)\rangle \,\mathrm{d}x = -\int_\Omega \eta(x)h(x)\mathrm{d}x \text{ for all } \eta \in C_c^\infty(\Omega),$$

in this case $\operatorname{div}(F) := h$.

Theorem: (Weak Gauß Theorem) If $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with C^1-boundary and if we have the function $F \in H^{\mathrm{div}}(\Omega)$ and the function $\eta \in C^\infty\left(\overline{\Omega}\right)$, then it holds

$$\int_\Omega \operatorname{div}(F)(x)\eta(x)\mathrm{d}x = -\int_\Omega \langle F(x), \nabla\eta(x)\rangle \,\mathrm{d}x + \int_{\partial\Omega} \langle \nu(x), F(x)\rangle \,\eta(x)\mathrm{d}s(x).$$