**Exercise 1 C**

Let \((E, \| \cdot \|)\) be a normed vector space. Prove that \(E\) is complete if and only if every absolutely convergent sequence is convergent, i.e., if, for \((x_n)_n \subset E,\)
\[
\sum_{n \in \mathbb{N}} \|x_{n+1} - x_n\| < \infty
\]
implies that \((x_n)_n\) converges.

**Exercise 2 C**

Let \(\Omega\) be a \(\sigma\)-finite measure space.

(a) Let \(f_1, f_2, \ldots, f_k\) be \(k\) functions such that \(f_i \in L^{p_i}(\Omega)\) \(\forall i = 1, \ldots, k\) with \(1 \leq p_i \leq \infty\) and \(\sum_{i=1}^{k} \frac{1}{p_i} \leq 1\). Set
\[
f(x) = \prod_{i=1}^{k} f_i(x).
\]
Show that \(f \in L^p(\Omega)\) with \(\frac{1}{p} = \sum_{i=1}^{k} \frac{1}{p_i}\) and that
\[
\|f\|_p \leq \prod_{i=1}^{k} \|f_i\|_{p_i}.
\]

(b) Show that if \(f \in L^p(\Omega) \cap L^q(\Omega)\) with \(1 \leq p, q \leq \infty\) then
\[
\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha},
\]
where
\[
\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad \text{and} \quad \alpha \in [0, 1].
\]

**Exercise 3**

Let \(\Omega\) be a \(\sigma\)-finite measure space and let \(0 < \alpha < 1\). Set
\[
L^\alpha(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } |u|^\alpha \in L^1(\Omega)\}
\]
and
\[
[u]_\alpha = \left( \int_\Omega |u|^\alpha \right)^{\frac{1}{\alpha}}.
\]
Show that \(L^\alpha(\Omega)\) is a vector space but \([\cdot]_\alpha\) is not a norm.

**Exercise 4**

(a) Let \(1 < p < \infty\). Prove that there is a constant \(C\) (depending only on \(p\)) such that
\[
|a - b|^p \leq C(|a|^p + |b|^p)^{1-\frac{q}{p}} \left( |a|^p + |b|^p - 2 \left| \frac{a + b}{2} \right|^p \right)^{\frac{q}{p}} \quad \forall a, b \in \mathbb{R}.
\]

(b) Let \(\Omega\) be a \(\sigma\)-finite measure space. Deduce that \(L^p(\Omega)\) is uniformly convex for \(1 < p \leq 2\).

*Hint for (b): Use (a) and Hölder’s inequality.*