Exercise 1 We have

\[ |u_n - u|^2 = \langle u_n - u, u_n - u \rangle = |u_n|^2 - 2\text{Re} \langle u_n, u \rangle + |u|^2. \]

By definition, \( u_n \rightharpoonup u \) means

\[ \lim_{n \to \infty} \langle u_n, u \rangle = \langle u, u \rangle = |u|^2. \]

On the other hand we have

\[ |u| \leq \liminf_{n \to \infty} |u_n| \leq \limsup_{n \to \infty} |u_n| \leq |u|, \]

i.e.,

\[ |u_n| \to |u|. \]

Thus

\[ |u_n - u|^2 \to 0, \]

i.e. \( u_n \to u \) in \( H \).

Exercise 2 Let \( a \in A \) be such that

\[ \|y - a\| = d := \min_{x \in A} \|y - x\| \]

and set

\[ b := y - a. \]

Now we show that \( b \in A^\perp \). So suppose, for contradiction, that \( b \notin A^\perp \). Then \( \beta := \langle b, x \rangle > 0 \) for some \( x \in A \). Clearly, \( x \neq 0 \) since otherwise \( \beta = 0 \). For any scalar \( \alpha \), we have

\[ \|y - (a + \alpha x)\|^2 = \|b - \alpha x\|^2 = \|b\|^2 - 2\bar{\alpha} \beta - |\alpha|^2 \|x\|^2. \]

Choosing \( \bar{\alpha} := \frac{\beta}{\|x\|^2} \), we get

\[ \|y - (a + \frac{\beta x}{\|x\|^2})\|^2 = \|b\|^2 - \frac{|\beta|^2}{\|x\|^2} < \|b\|^2 = \|y - a\|^2, \]

which is a contradiction to the fact that \( a \) is the minimizing vector of \( d \). Thus \( b \in A^\perp \).

To prove that this representation is unique, suppose that \( y \) can also be represented as

\[ y = a' + b' \quad \text{with} \quad a' \in A, b' \in A^\perp. \]

Then

\[ (a - a') + (b - b') = y - y = 0 \]

and

\[ (a - a') \perp (b - b') \quad \text{since} \quad a - a' \in A \quad \text{and} \quad b - b' \in A^\perp. \]

Pythagoras’ theorem then gives

\[ 0 = \|(a - a') + (b - b')\|^2 = \|a - a'\|^2 + \|b - b'\|^2, \]

implying that \( a = a' \) and \( b = b' \).