Exercise 1

Let \([a, b]\) be an interval and set

\[
\text{Lip}([a, b]) := \{ f : [a, b] \to \mathbb{R} : f \text{ is Lipschitz continuous} \}.
\]

Show that \(\text{Lip}([a, b])\) is a Banach space with respect to

\[
\| f \|_{\text{Lip}} := \| f \|_{\infty} + \sup_{x, y \in [a, b], x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.
\]

Furthermore, show that \(C^1([a, b]) \subset \text{Lip}([a, b]) \subset C([a, b])\) holds.

Exercise 2 C

(a) Let \((M, d)\) be a metric space. Show that

(i) \(d_1 := \min\{d, 1\}\),

(ii) \(d_2 := \frac{d}{1 + d}\)

define also metrics on \(M\).

(b) Consider \(d : \mathbb{R} \times \mathbb{R} \to [0, \infty), d(x, y) := |\arctan(x) - \arctan(y)|\). Show that \(d\) defines a metric on \(\mathbb{R}\). Is \((\mathbb{R}, d)\) complete?

Exercise 3 C

Consider \(X = C([0, 1])\) equipped with the \(\| \cdot \|_{\infty}\)-norm. Prove or disprove the following assertions:

(a) \(A := \{ f \in X : f([0, 1]) = [0, 1] \}\) is closed.

(b) \(B := \{ f \in X : f \text{ is injective} \}\) is closed.

(c) \(B\) is open.

(d) \(D := \{ f \in X : f\left(\frac{1}{2}\right) = 0 \}\) is open.