Exercise 1

Let $X$ be a Banach space. A linear operator $P : X \to X$ is called a projection if $P^2 = P$ (where $P^2 = P \circ P$). Show that $P$ is bounded if and only if $\ker P$ and $\operatorname{ran} P$ are closed.

Exercise 2

Let $(X, \| \cdot \|)$ be a Banach space and let $A, B$ be closed linear subspaces with $A \cap B = \{0\}$. Show that $A + B$ is closed if and only if there exists a constant $\alpha \geq 0$ such that

$$\| x \| \leq \alpha \| x + y \| \quad \forall x \in A, y \in B.$$  

Exercise 3 C

Let $X$ be a Banach space and $B \in L(X)$. Let $D(A) \subset X$ be a linear subspace and let $A : D(A) \to X$ be a closed linear operator. Show the following statements:

(a) The linear operator $AB : D(AB) \to X$, where $D(AB) = \{ x \in X : Bx \in D(A) \}$, is closed.

(b) The linear operator $BA : D(BA) \to X$, where $D(BA) = D(A)$, is closed if $B^{-1} \in L(X)$.

Remark: An operator is called closed if its graph is closed.

Exercise 4 C

Let $p \in (1, \infty)$, $(x_n)_n \subset l^p$ and $x \in l^p$. Show that $x_n \rightharpoonup x$ weakly in $l^p$ if and only if both of the following conditions are satisfied

(i) The sequence $(x_n)_n$ is bounded in $l^p$,

(ii) $x(j) = \lim_{n \to \infty} x_n(j) \forall j \in \mathbb{N}$. 