Exercise 3 Let $X$ be a Banach space, $B \in L(X), D(A) \subset X$ a linear subspace and let $A : D(A) \to X$ be a closed linear operator.

(a) Let $(x_n)_n \subset D(AB) = \{ x \in X : Bx \in D(A) \}$ be such that $(x_n, ABx_n) \to (x, y) \in X \times X$. We need to show that $x \in D(AB)$ and $y = ABx$.

Since $B \in L(X)$, we have $Bx_n \to Bx$. Furthermore, by the definition of $D(AB)$, we have $(Bx_n)_n \subset D(A)$. Since $A : D(A) \to X$ is closed, we get $Bx \in D(A)$ (i.e., $x \in D(AB)$) and $y = \lim_{n \to \infty} ABx_n = ABx$.

(b) Let $(x_n)_n \subset D(BA) = D(A)$ be such that $(x_n, BAx_n) \to (x, y) \in X \times X$. We need to show that $x \in D(BA)$ and $y = BAx$.

We have

$$Ax_n = B^{-1}BAx_n \to B^{-1}y$$

and using that $A$ is closed, we conclude that $x \in D(A) = D(BA)$ and $B^{-1}y = Ax$. Thus $y = BAx$.

Exercise 4 Assume that $x_n \to x$ in $l^p$. Then by Corollary 6.13, we get (i).

Fix $j \in \mathbb{N}$ and consider $f_j : l^p \to \mathbb{F}$ with $f_j(z) = z(j)$ This is an element of $(l^p)^*$. Hence

$$x(j) = f_j(x) = f_j(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f_j(x_n) = \lim_{n \to \infty} x_n(j),$$

i.e., (ii) holds.

Now assume that (i) and (ii) hold. Let $f \in (l^p)^*$. Then there exists $y \in l^q = (l^p)^*$ such that

$$f(z) = \sum_{j=1}^{\infty} z(j)y(j) \quad \forall z \in l^p.$$ 

Then

$$f(x) = \sum_{j=1}^{\infty} x(j)y(j) = \sum_{j=1}^{\infty} \lim_{n \to \infty} x_n(j)y(j).$$

If we manage to show that

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} x_n(j)y(j) = \sum_{j=1}^{\infty} x_n(j)y(j)$$

we are done, since then

$$f(x) = \lim_{n \to \infty} \sum_{j=1}^{\infty} x_n(j)y(j) = \lim_{n \to \infty} f(x_n).$$

To show [4], let $\varepsilon > 0$. Since $(x_n)_n$ is bounded in $l^p$ we get that $(x - x_n)_n$ is also bounded in $l^p$, i.e., there exists $A < \infty$ such that $\|x - x_n\|_p \leq A$ $\forall n \in \mathbb{N}$. Furthermore, since $y \in l^q$, there exists $K \in \mathbb{N}$ such that

$$\left( \sum_{j=K+1}^{\infty} |y(j)|^q \right)^{\frac{1}{q}} \leq \frac{\varepsilon}{2A}.$$
Thus we have

$$\left| \sum_{j=1}^{\infty} x(j)y(j) - \sum_{j=1}^{\infty} x_n(j)y(j) \right| \leq \left| \sum_{j=1}^{K} (x(j) - x_n(j))y(j) \right| + \left| \sum_{j=K+1}^{\infty} (x(j) - x_n(j))y(j) \right|$$

$$\leq \sum_{j=1}^{K} |x(j) - x_n(j)||y(j)| + \left( \sum_{j=K+1}^{\infty} |x(j) - x_n(j)|^p \right)^{\frac{1}{p}} \left( \sum_{j=K+1}^{\infty} |y(j)|^q \right)^{\frac{1}{q}}.$$ 

For every $j = 1, \ldots, K \exists N_j \in \mathbb{N}$ such that

$$|x(j) - x_n(j)||y(j)| \leq \frac{\varepsilon}{2K} \quad \forall n \geq N_j.$$

Set $N := \max_{j=1, \ldots, K} N_j$. Then

$$\left| \sum_{j=1}^{\infty} x(j)y(j) - \sum_{j=1}^{\infty} x_n(j)y(j) \right| \leq \frac{\varepsilon}{2} + A \frac{\varepsilon}{2A} = \varepsilon.$$