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Functional Analysis

1st Exercise Sheet

Exercise 1: Sequences
Prove the following claims:

1. Let (X, d) be a metric space, $(a_n)_{n\in\mathbb{N}}$ be a strictly positive decreasing null sequence in $\mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then there exists a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ such that
$$\forall n \in \mathbb{N} : d(x_{k(n+1)}, x_{k(n)}) \leq a_n \,.$$

2. Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X and $(x_{k(n)})_{n\in\mathbb{N}}$ be a subsequence such that $x_{k(n)} \xrightarrow{n\to\infty} x$. Then also $x_n \xrightarrow{n\to\infty} x$ holds.

3. Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then
$$x_n \xrightarrow{n\to\infty} x \Leftrightarrow \text{each subsequence of } (x_n)_{n\in\mathbb{N}} \text{ contains a subsequence converging to } x \,.$$

Exercise 2: Metric
Show that the following functions are metrics:

1. *French railway metric* The following function on $\mathbb{R}^2$:
$$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, (a, b) \mapsto d(a, b) = \begin{cases} \|a - b\|, & \text{if } a = cb \quad \text{for some } c \in \mathbb{R}\,, \\ \|a\| + \|b\|, & \text{otherwise}\,. \end{cases}$$

2. *Discrete metric* Let $X \neq \emptyset$. The following function on X:
$$d : X \times X \to \mathbb{R}, (x, y) \mapsto d(x, y) = \begin{cases} 0, & \text{if } x = y\,, \\ 1, & \text{otherwise}\,. \end{cases}$$