Institut für Analysis
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Functional Analysis

2nd Exercise Sheet

*** Exercise 3: Functions of Metric**
Let (X, d) be a metric space. Show that the following functions are also metrics on X:

(a) $d_a(x, y) = \frac{d(x,y)}{1+d(x,y)}$,

(b) $d_b(x, y) = \min\{d(x, y), 1\}$.

Furthermore show that the open sets w.r.t. (X, d_a) and (X, d_b) coincide with open sets w.r.t. (X, d).

*** Exercise 4: Completeness of Normed Spaces**
Prove or disprove completeness of the following spaces:

1. *Hölder spaces* Let $\alpha \in (0, 1)$. Space $C^\alpha([0, 1])$ consists of all functions $f : [0, 1] \to \mathbb{R}$ for which the following norm is finite
$$\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} .$$

2. *Continuous functions vanishing at infinity* $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) | \lim_{|x|\to\infty} f(x) = 0\}$with the supremum norm $\| \cdot \|_\infty$.

*** Exercise 5:**
Let $(X, \| \cdot \|)$ be a normed space and $M \neq \emptyset$ be an infinite set.

Summable family $(x_m)_{m\in M}$ *with value* $x \in X$
$\forall \epsilon > 0 \exists F \subset M \text{ finite} \forall G \supseteq F \text{ finite} : \left\| x - \sum_{m\in G} x_m \right\| \leq \epsilon$ with $x = \sum_{m\in M} x_m$.

Cauchy type condition for summability of family $(x_m)_{m\in M}$
$\forall \epsilon > 0 \exists F \subset M \text{ finite} \forall G \subset M \text{ finite}, F \cap G = \emptyset : \left\| \sum_{m\in G} x_m \right\| \leq \epsilon$.

Prove the following implication: X is a Banach space $\Rightarrow$ The above definitions are equivalent.

Exercise 6:
Let $(X, \|\cdot\|)$ be a Banach space, M be an infinite set and $Z := \left\{ (x_m)_{m\in M} \subset X \middle| (x_m)_{m\in M} \text{ is summable} \right\}$. Show that Z is a Banach space for the norm $\|(x_m)_{m\in M}\|_Z := \sup_{F\subseteq M,\text{finite}} \left\| \sum_{m\in F} x_m \right\|$.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.