Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

WS2018/19
7.11.2018

Functional Analysis

4th Exercise Sheet

* **Exercise 10: Separability**
Prove the following: Any subset of a separable metric space (X, d) is separable.

* **Exercise 11: Dual Spaces**
As in the lecture we denote for scalar sequences $x = (x_j)_{j\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{N}}$: $(Jy)(x) = \sum_{j=1}^{\infty} x_j y_j$ (provided that the series is convergent). Show the following:

(a) $J : l^1 \to (c_0)'$ is a linear isometric bijection.

(b) $J : l^\infty \to (l^1)'$ is a linear isometric bijection.

So, in this sense, the dual space of c_0 "is" l^1 and the dual space of l^1 "is" l^∞.

* **Exercise 12: Dual Space**
Let $\Omega = \mathbb{N}$ and denote by μ the measure on $\mathfrak{P}(\mathbb{N})$ induced by

$$\mu(\{n\}) = \begin{cases} 1 & n \text{ even} \\ \infty & n \text{ odd} \end{cases}$$

Determine the space $\mathscr{L}^1(\Omega, \mu)$ and find a description of its dual space.

* **Exercise 13: σ-algebra matters**
Let $\Omega = \mathbb{R}$, $\Sigma_0 = \{A | A \text{ countable or } \mathbb{R} \setminus A \text{ countable}\}$ and $\Sigma_1 = \mathfrak{P}(\mathbb{R})$. For $j = 0, 1$ we denote by $B(\Omega, \Sigma_j)$ the space of all bounded Σ_j-measurable functions $f : \Omega \to \mathbb{K}$ equipped with the sup-norm. For j=0,1 define $\mu_j : \Sigma_j \to [0, \infty]$ by

$$\mu_j(A) = \begin{cases} |A| & A \text{ finite} \\ \infty & \text{otherwise} \end{cases}$$

Show the following

(1) $\mathscr{L}^1(\Omega, \mu_0) = \mathscr{L}^1(\Omega, \mu_1)$,
Hint: Show that $f \in \mathscr{L}^1(\Omega, \mu_j)$ if and only if $\Sigma_{x\in S_f} |f(x)| < \infty$ where $S_f := \{x | f(x) \neq 0\}$ is countable.

(2) $B(\Omega, \Sigma_0)$ is a proper subset of $B(\Omega, \Sigma_1)$ and

(3) The dual space of $\mathscr{L}^1(\Omega, \mu_0)$ can be identified with $B(\Omega, \Sigma_1)$.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.