Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

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Functional Analysis

5th Exercise Sheet

*** Exercise 14: Multiplication Operator**
Let $\Omega = \mathbb{R}$, $\Sigma = \mathfrak{P}(\mathbb{R})$ and

$$\mu(A) = \begin{cases} |A| & A \subset \mathbb{Q} \\ \infty & \text{otherwise}. \end{cases}$$

We define an operator on $X = \mathscr{L}^1(\Omega, \mu)$ as $M : X \to X$, $(M\psi)(x) = \theta(x)\psi(x)$ where $\theta \in B(\Omega, \Sigma)$. Show that $\|M\| \le \|\theta\|_\infty$ and that the equality holds if $\|\theta\|_\infty = \|\theta 1_A\|_\infty$ for a σ-finite subset A.

*** Exercise 15: Embeddings for Spaces of Sequences**
Let $1 \le p \le q < \infty$. Show the following:

(a) l^p is a subset of l^q. *Hint*: Show that $\|x\|_q \le \|x\|_p \ \forall x \in l^p$ and start with $\|x\|_p = 1$.

(b) $\bigcup_{p<\infty} l^p$ is a proper subset of c_0.

(c) For every $x \in l^1$ we have $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

Exercise 16: Sequence Product
Let $p, q, r \in [1, \infty)$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $x \in l^p$, $y \in l^q$, $z \in l^r$ and define xyz pointwise, i.e. $(xyz)_n = x_n y_n z_n$. Then $xyz \in l^1$ with $\|xyz\|_1 \le \|x\|_p \|y\|_q \|z\|_r$.

*** Exercise 17: Embeddings for L^p Spaces**
Let $a, b \in \mathbb{R}$ and $1 \le p \le q < \infty$. Show the following

(1) $L^q([a,b]) \subset L^p([a,b])$. *Hint*: Show that $\frac{\|\psi\|_p}{(b-a)^{1/p}} \le \frac{\|\psi\|_q}{(b-a)^{1/q}} \ \forall \psi \in L^q([a,b])$.

(2) Let I be an unbounded interval. Then for the case $p \ne q$ neither $L^q(I) \subset L^p(I)$ or $L^p(I) \subset L^q(I)$ hold.

Exercise 18: Embeddings for L^p Spaces
Let $a, b \in \mathbb{R}$ and $1 \le p < \infty$. Show the following

(1) $L^\infty([a,b]) \subset L^p([a,b])$. *Hint*: Show that $\|\psi\|_p \le (b-a)^{1/p} \|\psi\|_\infty \ \forall \psi \in L^\infty([a,b])$.

(2) $L^\infty([a,b]) \subset \bigcap_{p<\infty} L^p([a,b])$. Furthermore subset is proper.

(3) For the case of whole $\mathbb{R}$ can not either $L^\infty(\mathbb{R}) \subset L^p(\mathbb{R})$ or $L^p(\mathbb{R}) \subset L^\infty(\mathbb{R})$ hold.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.