Institut für Analysis
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Functional Analysis

6th Exercise Sheet

Exercise 19: Sum of Banach Spaces
Let Z be $\mathbb{K}$ vector space with a metric such that the mapping $U : (Z, d) \to (Z, d) : z \mapsto -z$ is continuous, $X, Y \subseteq Z$ be linear subspaces and $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces. Furthermore metric d on Z is such that identity mappings $I : (X, \|\cdot\|_X) \to (Z, d)$ and $I : (Y, \|\cdot\|_Y) \to (Z, d)$ are continuous. Show that the vector space $(X + Y, \|\cdot\|)$ is a Banach space with the norm defined as

$$\|z\|_{X+Y} := \inf\{\|x\|_X + \|y\|_Y | x \in X, y \in Y, x + y = Z\}.$$

Hint: Represent $X + Y$ as a quotient space of $X \otimes Y$.

* **Exercise 20: General Hölder Inequality**
Proof the following claim: Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $(f, g) \mapsto fg$ is continuous $L^p(\Omega, \mu) \times L^q(\Omega, \mu) \to L^r(\Omega, \mu)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$.

* **Exercise 21: Equivalent Norms**
Let K be a compact metric space, $\omega \in C(K)$ and $\omega > 0$. Show that the following norm is equivalent to the supremum norm

$$\|f\|_\omega := \sup_{x \in K} (|f(x)|\omega(x)).$$

* **Exercise 22: Compact Sets in l^2**
In this exercise we consider the l^2 space of real sequences. Check the following subsets for boundedness and compactness:

(1) $E_1 := \left\{x \in l^2 \middle| |x_n| \leq \frac{1}{\sqrt{n}} \text{ for all } n\right\}$,

(2) $E_2 := \left\{x \in l^2 \middle| \sum_{n=1}^{\infty} |x_n|^2 \leq 1\right\}$,

(3) $E_3 := \left\{x \in l^2 \middle| |x_n| \leq \frac{1}{n} \text{ for all } n\right\}$ *(Hilbert's cube)*,

(4) $E_4 := \left\{x \in l^2 \middle| \exists y \in l^2 : \|y\|_2 \leq 1, |x_n| \leq \frac{|y_n|}{n} \text{ for all } n\right\}$.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.