Karlsruher Institut für Technologie

Institut für Analysis
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Functional Analysis

7th Exercise Sheet

Exercise 23:
Let X be a Banach space and Y be separable Banach space with the following property: There is a sequence of finite-rank operators $S_n \subseteq F(Y)$ such that

$$\lim_{n\to\infty} S_n y = y, \quad \forall y \in Y\,.$$

Then the following holds $\overline{F(X,Y)} = K(X,Y)$.
Hint: Show $\|S_n T - T\| \to 0$. Start by covering $T(B_X)$ with finitely many balls of radius $\epsilon > 0$.

* **Exercise 24:**
Let $1 \le p < \infty$ and $A \subseteq l^p$. Then the following are equivalent:

1. A is relatively compact
2. A is bounded and $\lim_{n\to\infty} \sup_{x\in A} \left(\sum_{j=n}^{\infty} |x_j|^p\right)^{\frac{1}{p}} = 0$

Exercise 25:
Consider $C^\alpha([0,1])$ with $0 < \alpha < 1$ and norm $\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$. Show that the identity $I : (C^\alpha([0,1]), \|\cdot\|_{C^\alpha}) \to (C([0,1]), \|\cdot\|_\infty)$ is compact.

* **Exercise 26:**
Let $\omega := \mathbb{K}^{\mathbb{N}}$ be the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$. Show that ω is a complete metric space for

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

and

$$\lim_{n\to\infty} d(x^{(n)}, x^{(0)}) = 0 \Leftrightarrow \forall j \in \mathbb{N} : \lim_{n\to\infty} x_j^{(n)} = x_j^{(0)}\,.$$

* **Exercise 27:**
Let X be a Banach space with $\dim X = \infty$. Then no basis of X is countable.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.