Institut für Analysis
PD Dr. Peer Christian Kunstmann
Dr. Michal Jex

WS2018/19
19.12.2018

Functional Analysis

10th Exercise Sheet

* **Exercise 36:**
Let X be a normed space, $V_1, V_2 \subseteq X$ convex, $V_1 \cap V_2 = \emptyset$, V_1 closed and V_2 compact. Show that there exists $x' \in X'$ such that

$$\sup\{\operatorname{Re} x'(v_2) : v_2 \in V_2\} < \inf\{\operatorname{Re} x'(v_1) : v_1 \in V_1\}.$$

Hint: Proceed as in the proof of Thm 3.14 from Lecture notes.

* **Exercise 37:**
Show that $(BV_0[a,b], \|\cdot\|_{BV})$ is a Banach space.

* **Exercise 38:**
Let X be Banach space and $Y \subseteq X$ be a closed linear subspace. Show that the following are equivalent

1. There exists $P \in \mathfrak{L}(X), P^2 = P$ such that $P(X) = Y$.
2. There exists a closed linear subspace $Z \subseteq X$ such that $Z \cap Y = \{0\}$, $Z + Y = X$. (In this case $U : Y \times Z \to X, (y,z) \mapsto y + z$ is an isomorphism. Why?)
3. Mapping $T_0 : Y \to Y \times X/Y, y \mapsto (y, Y)$ has an extension to an isomorphism $T : X \to Y \times X/Y$.
4. There exists a selection operator $Q \in \mathfrak{L}(X/Y, X)$, i.e. Q is injective, $Q(u) \in u$ for all $u \in X/Y$ and $Q^{-1} : Q(X/Y) \to X/Y$ is continuous.

Exercise 39: Generalization of limits in c to l^∞
For $x \in l^\infty$ we define the following

$$\pi(x; n_1, \ldots, n_k) = \limsup_{n\to\infty} \frac{1}{k} \sum_{j=1}^{k} x_{n+n_j}$$

and

$$q(x) = \inf \pi(x; n_1, \ldots, n_k)$$

where the infimum is taken over all k-tuples of natural numbers $n_1, \ldots, n_k$. Show the following

1. q is a positive homogeneous and subadditive functional.
2. There exists a linear functional $u(x)$ such that $u(x) \le q(x)$ for all $x \in l^\infty$. We set $\operatorname{Lim}_{n\to\infty} x_n := u(x)$.
3. If $x \in c$ then $\lim_{n\to\infty} x_n = \operatorname{Lim}_{n\to\infty} x_n$.

We wish you Merry Christmas and Happy New Year

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.