Functional Analysis

11th Exercise Sheet

* Exercise 40: Lax-Milgram Lemma

Let $\beta : H \times H \to \mathbb{K}$ be a sesquilinear form which is continuous, i.e. there exists $M > 0$ such that $\forall x, y \in H : |\beta(x, y)| \le M\|x\|\|y\|$. Show the following

1. There exists a uniquely defined operator $B \in \mathscr{L}(H)$ with $\forall x, y \in H : (Bx|y) = \beta(x, y)$.
2. Assume that β is coercive, i.e. there exists $\eta > 0$ such that $\forall x \in H : \operatorname{Re} \beta(x, x) \ge \eta\|x\|^2$. Then $B : H \to H$ is an isomorphism and $\|B^{-1}\| \le \eta^{-1}$.

* Exercise 41:

Let X be a real Banach space. A closed halfspace in X has the form $H(x', \alpha) := \{x \in X | x'(x) \le \alpha\}$ where $x' \in X' \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Assume that $\emptyset \ne K \subseteq X$ is closed and convex. Show that

$$K = \bigcap \left\{ H(x', \alpha) | x' \in X' \setminus \{0\}, \alpha \in \mathbb{R}, K \subseteq H(x', \alpha) \right\} .$$

Exercise 42: Gram-Schmidt process

Let H be a Hilbert space. Show the following

1. Let $Y \subseteq H$ be a closed linear subspace which is separable. Assume that $(e_n)_{n=1}^N$, where $N \in \mathbb{N} \cup \{\infty\}$ is an ONB of Y. Then $P_Y x = \sum_{n=1}^N (x|e_n)e_n$.
2. Let $Y_k = \operatorname{lin}\{x_1, \dots, x_k\}$ then we can construct orthonormal set $S = \operatorname{lin}\{e_1, \dots, e_k\}$ such that $\operatorname{lin} S = Y_k$ and $f_{j+1} = x_{j+1} - \sum_{l=1}^j (x_{j+1}|e_l)e_l$, $e_{j+1} = \frac{f_{j+1}}{\|f_{j+1}\|}$ for $\|f_{j+1}\| \ne 0$.

* Exercise 43:

Let $H = L^2([-1, 1])$ and $P_n(t) = \frac{1}{2^n n!} \left(\frac{d}{dt}\right)^n (t^2 - 1)^n$, $n \in \mathbb{N}_0$ be Legendre polynomials. Show the following properties of $P_n(t)$ for all $n \in \mathbb{N}$

1. $P_n(1) = 1$,
2. $P'_{n+1}(t) = \frac{1}{2^n n!} \frac{d^{n+1}}{dt^{n+1}} \left(t(t^2 - 1)^n\right)$,
3. $P'_{n+1}(t) = (2n + 1)P_n(t) + P'_{n-1}(t)$,
4. $P'_{n+1}(t) = tP'_n(t) + (n + 1)P_n(t)$,
5. $nP_n(t) = tP'_n(t) - P'_{n-1}(t)$,
6. $(1 - t^2)P'_n(t) = nP_{n-1}(t) - ntP_n(t)$,
7. $Q_n(t) := \frac{d}{dt}\left((1 - t^2)P'_n(t)\right) + n(n + 1)P_n(t) = 0$,
8. for $n \ne m$ $P_n \perp P_m$ in H,
9. $R_n(t) := (n + 1)P_{n+1}(t) - (2n + 1)tP_n(t) + nP_{n-1}(t) = 0$,
10. $\int_{-1}^1 P_n(t)^2 dt = \frac{2n-1}{2n+1} \int_{-1}^1 P_{n-1}(t)^2 dt$ for $n \ge 2$
11. $e_n(t) = \sqrt{n + \frac{1}{2}} P_n(t), n \in \mathbb{N}$ is orthonormal set obtained by Gram-Schmidt process on, $x_n(t) = t^n$ for all $n \in \mathbb{N}_0$.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.