Institut für Analysis
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WS2018/19
16.1.2019

Functional Analysis

12th Exercise Sheet

Exercise 44:
For $\xi \in \mathbb{R}$ let $e_\xi : \mathbb{R} \to \mathbb{C}, e_\xi(t) := e^{i\xi t}$ and let $X = \text{lin}\{e_\xi | \xi \in \mathbb{R}\}$. Show that

$$(f|g) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\overline{g(t)}dt$$

defines a scalar product on X. Let $AP^2(\mathbb{R})$ denote the completion of X.

1. Find an ONB in $AP^2(\mathbb{R})$ and show that $AP^2(\mathbb{R})$ is not separable.
2. Show that $AP^2(\mathbb{R})$ contains all functions in $\overline{X}^{\|\cdot\|_\infty}$
(where the closure is calculated in $(BUC(\mathbb{R}), \|\cdot\|_\infty)$).

* **Exercise 45:**
Let H be a Hilbert space and $K \subseteq H$ be a closed convex set. Show that

$$\|P_K x - P_K y\| \leq \|x - y\|$$

for all $x, y \in H$ where P_K denotes projection onto set K as defined in Remark to Proposition 4.3 in Lecture notes.

* **Exercise 46:**
1. Let $U, V \subseteq \mathbb{R}^n$ be bounded and open sets such that $\overline{U} \subseteq V$. Show that there exists a function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for all $x \in U$ and $\varphi(y) = 0$ for all $y \in V^c$.
2. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Show that there exists a sequence of functions $\psi_k \in C_c^\infty(\Omega)$ such that $\sum_k \psi_k = 1$ on Ω and the functions have finite overlap, i.e. $\sum_k 1_{\text{supp}\,\psi_k} \leq C$.

* **Exercise 47:**
1. Let $p, q \in [1, \infty)$. Prove a product rule for weak derivatives, i.e. show that

$$\partial_j(fg) = g\partial_j f + f\partial_j g$$

for $f \in W^{1,p}(\mathbb{R}^n), g \in W^{1,q}(\mathbb{R}^n)$ and $\frac{1}{p} + \frac{1}{q} \leq 1$.
2. Let $M : W^{m,p}(\mathbb{R}^n) \times W^{m,q}(\mathbb{R}^n) \to W^{m,r}(\mathbb{R}^n)$ be a mapping defined as $(f, g) \mapsto fg$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Show that the mapping is continuous.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.