Institut für Analysis
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Functional Analysis

13th Exercise Sheet

Exercise 48:
Let $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$ where I is an interval. Then there exists a function $\tilde{u} \in C\left(\overline{I}\right)$ such that

$$u = \tilde{u} \quad \text{a.e. on } I$$

and

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)dt \quad \forall x, y \in \overline{I}\,.$$

* **Exercise 49:**
Let $H = L^2(\Omega)$ be a Hilbert space of real valued functions. We define following sets

$$K_1 = \{f \in H | f \geq 0 \ a.e.\}\,, \quad K_2 = \{f \in H | f \leq 1 \ a.e.\}\,, \quad K_3 = \{f \in H | 0 \leq f \leq 1 \ a.e.\}\,.$$

1. Show that $P_{K_1} f = f_+ = \max\{f, 0\}$ and $P_{K_2} f = \min\{f, 1\}$.
2. Find the expression for P_{K_3}.

* **Exercise 50:**

1. Show that $\varphi \in C_c^1(\Omega)$ can be approximated by $C_c^\infty(\Omega)$ functions in C^1-norm.
2. Let $f \in L^1_{loc}(\Omega)$ has first weak derivatives everywhere. Show that $\int_\Omega f \partial_j \varphi dx = -\int_\Omega (\partial_j f)\varphi dx$ for all $\varphi \in C_c^1(\Omega)$.

* **Exercise 51:**
Let $a : \Omega \to \mathbb{C}^{n \times n}$ be a measurable and essentially bounded such that there exists $\eta > 0$ $\mathrm{Re}\left(\sum_{j,k=1}^n a_{jk}(x)\xi_k \overline{\xi}_j\right) \geq \eta |\xi|^2$ a.e. for all $\xi \in \mathbb{C}^n$. Furthermore let $\mathfrak{a} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$:

$$\mathfrak{a}(u, v) = \int_\Omega \sum_{j,k=1}^n a_{jk}(x) \partial_k u(x) \overline{\partial_j v(x)} dx\,.$$

Show that for any $f \in L^2(\Omega)$ the weak formulation of the problem

$$\begin{cases} -div(a(x)\nabla u) &= f \quad \text{in } \Omega \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

has an unique solution in $H_0^1(\Omega)$.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.