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# Functional Analysis

## 15th Exercise Sheet

**Exercise 56:**
Let $X$, $Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$. Let $x_n$ be a weakly convergent sequence in $X$. Then

$$x_n \rightharpoonup x_0 \,\text{in}\, X \Rightarrow Tx_n \rightharpoonup Tx_0 \,\text{in}\, Y\,.$$

**Exercise 57:**
Let $X$, $Y$ be Banach spaces and $T : X \supseteq D(T) \to Y$ linear. Then

$$T \,\text{is closed} \Leftrightarrow T \,\text{is weakly closed}\,.$$

*Definition*: $T$ is weakly closed:$\Leftrightarrow$ if $x_n \rightharpoonup x_0$ in $X$, $Tx_n \rightharpoonup y \in Y$ then $Tx_0 = y$.

**Exercise 58:**
Let $X$ be a reflexive Banach space, $K \subseteq X$ closed and convex and $f : K \to \mathbb{R}$ be continuous, convex function. Furthermore for the case that $K$ is unbounded assume also

$$f(x) \to \infty \,\text{for all}\, x \in \mathbb{K} \,\text{such that}\, \|x\| \to \infty\,.$$

Then $f$ attains its Minimum in $K$.

**Exercise 59:**
Let $X$ be a normed space and $K \subseteq X$ be a subset. Show the following

$$K \,\text{is bounded}\, \Leftrightarrow K \,\text{is weakly bounded}\,.$$

**Exercise 60:**
Let $f \in L^\infty(\mathbb{R}^n)$. Show the following

1. $\tau_y f := f(\cdot - y)$ tends to $f$ as $y \to 0$ in a weak-$\star$ sense, i.e. $\langle g, \tau_y f\rangle \to \langle g, f\rangle$ for all $g \in L^1(\mathbb{R}^n)$.

2. Let $\varphi_k$ be a mollifier as in 5.2. Then $\varphi_k \star f$ tends to $f$ as $k \to \infty$ in a weak-$\star$ sense.

**Remark:** There are no exercises which can be handed in this week.