Institut für Analysis
Institut für Analysis
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Functional Analysis

Final exam

Name	Student number

Points					
Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Total

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Exercise 1: (10 Points)
Let $a, b \in \mathbb{R}$ and $1 \leq p < \infty$. Show the following

(a) $L^\infty([a,b]) \subseteq L^p([a,b])$.
Hint: Show that $\|\psi\|_p \leq (b-a)^{1/p}\|\psi\|_\infty \ \forall \psi \in L^\infty([a,b])$.

(b) $L^\infty([a,b])$ is a proper subset of $\bigcap_{p<\infty} L^p([a,b])$.

(c) Neither $L^\infty(\mathbb{R}) \subseteq L^p(\mathbb{R})$ nor $L^p(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ does hold.

Exercise 2: (10 Points)
Let $T : C([0,3]) \to C([1,2]), f \mapsto f|_{[1,2]}$. Find $S : C([1,2]) \to C([0,3])$ such that $TS = I_{C([1,2])}$.
Show the following:

(a) $R(S)$ is closed.

(b) $R(S) \oplus N(T) = C([0,3])$.

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Exercise 3: (10 Points)
Let $T \in \mathcal{L}(c_0, l^\infty)$. Assume that, for each $x \in c_0$, there exists $M, k \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} : |(Tx)_n| \leq Mn^{-k},$$

where $(Tx)_n$ denotes the n-term of the sequence Tx. Show that there exist $\tilde{M}, k \in \mathbb{N}$ such that

$$\forall x \in c_0 \forall n \in \mathbb{N} : |(Tx)_n| \leq \tilde{M} n^{-k} \|x\|_\infty$$

Hint: Apply Baire's category theorem in c_0.

Exercise 4: (10 Points)
Let $(h_n) \in \mathbb{R} \setminus \{0\}$ such that $h_n \to 0$ be fixed.

(a) For all $\varphi \in C_c^\infty(\mathbb{R})$ show

$$\frac{\varphi(\cdot - h_n) - \varphi}{h_n} \to -\varphi' \quad \text{in} \quad \| \cdot \|_{L^2(\mathbb{R})} .$$

(b) Let $f \in L^2(\mathbb{R})$, $C > 0$ and

$$\|f(\cdot + h_n) - f\| \leq C|h_n| \quad \text{for all}\, n .$$

Show that $f \in W^{1,2}(\mathbb{R})$.
Hint: Use weak convergence.

Exercise 5: (10 Points)
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $u_0 \in H^1(\Omega)$, $f \in L^2(\Omega)$. Show that

$$\begin{aligned} \Delta u &= f \quad \text{in}\, \Omega , \\ u - u_0 &\in H_0^1(\Omega) \end{aligned}$$

there is a unique solution $u \in H^1(\Omega)$ of the weak formulation of the problem.

Hint 1: The weak interpretation of Δu is $\text{div}(\nabla u)$. Recall $\text{div} F \in H^{-1}(\Omega) = (H_0^1(\Omega))^\star$.
Hint 2: Write $u = v + u_0$ where $v \in H_0^1(\Omega)$. Find the weak formulation of the problem for v.

Good luck!