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# Functional Analysis

Test exam

**Exercise 1:** (10 Points)
Let $H$ be Hilbert space and $S := \{e_n | n \in \mathbb{N}\} \subseteq H$ be an orthonormal set. Show that the following are equivalent:

(a) $\text{lin}S$ is dense in $H$.

(b) If $x \in H$ satisfies $x \perp S$ then $x = 0$.

(c) $x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$ for all $x \in H$.

**Exercise 2:** (10 Points)
Let $1 \leq p \leq q < \infty$. Show the following:

(a) $l^p$ is a subset of $l^q$.

(b) $\bigcup_{p<\infty} l^p$ is a proper subset of $c_0$.

(c) For every $x \in l^1$ we have $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

**Exercise 3:** (10 Points)
Let $X$ be a Banach space and $M \subseteq X$. Let $M$ be weakly bounded, i.e., for any $\Phi \in X'$ the set $\{\langle x, \Phi\rangle : x \in M\} \subseteq \mathbb{K}$ is bounded. Then $M$ is strongly bounded, i.e., the set $\{\|x\| : x \in M\} \subseteq \mathbb{R}$ is bounded.

**Exercise 4:** (10 Points)
Let $X$ be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence in $X$ with $\|x_n\|_X \leq 1$. Let

$$Y := \left\{ y \in X \middle| \exists (\lambda_n)_{n\in\mathbb{N}} \in l^1 : y = \sum_{n=1}^{\infty} \lambda_n x_n \ (\text{convergence in } \|\cdot\|_X) \right\}.$$

For $y \in Y$ we introduce

$$\|y\|_Y := \inf \left\{ \sum_{n=1}^{\infty} |\lambda_n| \middle| (\lambda_n)_{n\in\mathbb{N}} \in l^1 : y = \sum_{n=1}^{\infty} \lambda_n x_n \right\}.$$

Show the following:

(a) $(Y, \|\cdot\|_Y)$ is a normed space,

(b) $(Y, \|\cdot\|_Y)$ is complete.

**Exercise 5:** (10 Points)
Consider the function

$$u(x) = (1 + x^2)^{-\frac{\alpha}{2}} (\log(2 + x^2))^{-1}, \quad x \in \mathbb{R}$$

with $0 < \alpha < 1$. Check that $u \in W^{1,p}(\mathbb{R})$ for every $p \in [1/\alpha, \infty]$ and that $u \notin L^q(\mathbb{R})$ for every $q \in [1, 1/\alpha)$.

## Good luck!