10) (X,d) separable $\Rightarrow \exists E$ dense, countable in X ; M subset of X

$\forall x \in X \ \exists e_n \in E : d(e_n, x) < \frac{1}{n}$

for every e in E we have $d(e,M) = \inf\{d(e,m) \mid m \in M\}$

we find for every e in E choose $a_n \in M$ such that $d(a_n, e) < d(e,M) + \frac{1}{n}$

let A be set of all $a_n \Rightarrow A$ countable and $A \subset M$

now show A dense in M

$\varepsilon > 0 : \exists e \in E \quad d(m,e) < \frac{\varepsilon}{3} \quad m \in M \Rightarrow d(e,M) \leq \frac{\varepsilon}{3}$

$\exists a \in A: d(e,a) < d(e,M) + \frac{\varepsilon}{3} \leq \frac{2\varepsilon}{3}$

$d(m,a) \leq d(m,e) + d(e,a) < \varepsilon$

$\Rightarrow A$ dense in M

11) a) we want to show 1) J functional, 2) J linear, 3) J bijective, 4) J isometric $\|J(y)\|_{c_0'} = \|y\|_1$

ad 1) $|(J_y)(x)| = \left|\sum_{j=1}^{\infty} x_j y_j\right| \le \|x\|_\infty \|y\|_1 \quad \forall x \in c_0$

$\Rightarrow J_y \in (c_0)', \ \|J_y\|_{(c_0)'} \le \|y\|_1$

ad 2) $(J_y)(\alpha a + b) = \sum_{j=1}^{\infty} y_j (\alpha a + b)_j = \alpha (J_y)(a) + (J_y)(b)$ provided that RHS converge

from 1),+2) $l_1 \subseteq c_0^*$

ad 3) $\phi \in (c_0)', \ y_j := \phi(e_j)$ $\quad (e_j)_k = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}$ $\quad x \in c_0 \quad x = \sum_j x_j e_j \quad x_j \to 0$

$\phi(x) = \sum_j x_j \phi(e_j) = \sum_j x_j y_j \qquad |\phi(x)| \le \left|\sum_j x_j \phi(e_j)\right|$

$\Rightarrow \phi$ determined by y_j and $|\phi(x)| \le \|x\|_\infty \|y\|_1 \Rightarrow \|\phi\| \le \|y\|_1$

ad 4) we need $\|y\|_1 \le \|\phi\|$ which along with $\|\phi\| \le \|y\|_1$ completes the proof

$\sum_{j=1}^{n} |y_j| \underset{\uparrow}{=} \left|\sum_{j=1}^{n} x_j y_j\right| = |\phi(x)| \le \|\phi\| \underbrace{\|x\|_\infty}_{=1} \Rightarrow y \in l^1$ and $\|y\|_1 \le \|\phi\|$

choose $x_j = \operatorname{sgn} y_j \begin{cases} \frac{y_j}{|y_j|} & y_j \ne 0 \\ 0 & \text{otherwise} \end{cases}$

b) show 1) J functional, 2) J linear, 3) J bijective, 4) J isometric $\|J(y)\|_{(l^1)'} = \|y\|_\infty$

ad 1) $|(J_y)(x)| = \left|\sum_{j=1}^{\infty} x_j y_j\right| \le \|x\|_{l^1} \|y\|_\infty \quad \forall x \in l^1$

$\Rightarrow J_y \in (l^1)', \ \|J_y\|_{(l^1)'} \le \|y\|_\infty$

ad 2) $(J_y)(\alpha a + b) = \sum_{j=1}^{\infty} y_j (\alpha a + b)_j = \alpha (J_y)(a) + (J_y)(b)$ provided that RHS converge

ad 3) $\phi \in (l^1)', \ y_j := \phi(e_j) \qquad x \in l^1 \qquad x = \sum_j x_j e_j$ x converge in l^1

$\phi(x) = \sum_j x_j \phi(e_j) = \sum_j x_j y_j$ converge in $\mathbb{K}$

$\Rightarrow \phi$ determined by y_j and $|\phi(x)| \le \|y\|_\infty \|x\|_1 \Rightarrow \|\phi\| \le \|y\|_\infty$

ad 4) we need $\|y\|_\infty \le \|\phi\|$

$|y_j| = \left|\sum_{j=1}^{\infty} x_j y_j\right| = |\phi(x)| \le \|\phi\| \underbrace{\|x\|_1}_{=1}$

$x_k = \begin{cases} 0 & \text{otherwise} \\ \operatorname{sgn} y_j & j = k \end{cases}$ $\quad$ we can write this for every non-zero functional $\quad$ =1 for k big enough

12) $\mathcal{L}^1(\Omega,\mu) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \int |x_n| \, d\mu = \sum_{n=1}^{\infty} |x_n| \mu(\{n\}) < \infty \right\}$

$$\mathcal{L}^1(\Omega,\mu) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; x_{2k-1} = 0,\ k\in\mathbb{N} \text{ and } \sum_{n=1}^{\infty} |x_{2n}| < \infty \right\}$$

$$\mathcal{L}^\infty(\Omega,\mu) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; x_{2k-1} = 0,\ k\in\mathbb{N} \text{ and } \sup_n x_n < \infty \right\} = \left[\mathcal{L}^1(\Omega,\mu)\right]'$$

proof of dual space works in the same way as in 11(a)

13) 1) $f \in \mathcal{L}^1(\Omega, \mu_0)$:

$\Leftarrow$: $\sum_{x \in S_f} |f(x)| < \infty \Rightarrow \sum_{n=1}^{\infty/N} |f(x_n)| \, d\mu(\{x_n\}) < \infty \Rightarrow f \in \mathcal{L}^1(\Omega, \mu_0)$

$\Rightarrow$: by contradiction: $\infty = \sum_{n=1}^{\infty/N} |f(x_n)| = \sum_{n=1}^{\infty} |f(x_n)| \, d\mu(\{x_n\}) \Rightarrow f \notin \mathcal{L}^1(\Omega, \mu_0)$ *

$f \in \mathcal{L}^1(\Omega, \mu_1)$:

$\Leftarrow$: $\sum_{x \in S_f} |f(x)| < \infty \Rightarrow \sum_{n=1}^{\infty/N} |f(x_n)| \, d\mu(\{x_n\}) < \infty \Rightarrow f \in \mathcal{L}^1(\Omega, \mu_0)$

* or from lecture $f \in \mathcal{L}^p(\Omega, \mu)$, $p \in [1, \infty) \Rightarrow$ $\Rightarrow \{f \neq 0\}$ σ-finite

$\Rightarrow$ by contradiction: $\infty = \sum_{n=1}^{\infty} |f(x_n)| = \sum_{n=1}^{\infty} |f(x_n)| \, d\mu(\{x_n\}) \Rightarrow f \notin \mathcal{L}^1(\Omega, \mu_1)$ *

2) $\Sigma_0 \subset \Sigma_1 \Rightarrow B(\Omega, \Sigma_0) \subseteq B(\Omega, \Sigma_1)$

to show that it is proper subset means that $\exists f \in B(\Omega, \Sigma_1)$ such that $f \notin B(\Omega, \Sigma_0)$

measurable function $\Leftrightarrow$ preimage of measurable set is measurable

in $\mathbb{K}$ standard Borel σ-algebra (algebra generated by open sets)

$$f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1+x & x \in (0, 1) \\ 0 & x \in [1, \infty) \end{cases}$$

this function in $B(\Omega, \Sigma_1)$ but not in $B(\Omega, \Sigma_0)$ because preimage of measurable set $(1,2)$ is not measurable in Σ_0

in general choose uncountable set with uncountable complement

3) we introduce $(J\varphi)(\psi) = \int \varphi \psi \, d\mu_0$ and show that $J: B(\Omega, \Sigma_1) \to \mathcal{L}^1(\Omega, \mu_0)'$ is lin bijection

1) $|(J\varphi)(\psi)| = |\int \varphi \psi \, d\mu| \leq \|\varphi\|_\infty \|\psi\|_1 \Rightarrow J\varphi \in (\mathcal{L}^1(\Omega, \mu_0))'$ $\|J\varphi\| \leq \|\varphi\|_\infty$

2) $J\varphi$ linear ✓

3) we take $\phi \in (\mathcal{L}^1(\Omega, \mu_0))'$ $\phi(1_{x_j}) = \psi(x_j)$ $\varphi = \sum_{j=1}^{N} s_j 1_{A_j}$

$\phi(\varphi) = \sum_{j=1}^{N} s_j \phi(A_j) = \sum s_j \psi(x_j)$

$\Rightarrow \phi$ determined by values for every x_j and $\|\phi\| \leq \|\psi\|_\infty \|\varphi\|_1$

4) $|\psi(x_j)| \leq \|\phi\|$

$|\psi(x_j)| \leq |\sum_{j=1}^{n} \psi(x_j) \varphi(x_j)| = |\phi(\varphi)| \leq \|\phi\| \|\varphi\|_1$

$$\varphi(x_k) = \begin{cases} \operatorname{sgn} \psi(x_k) & x_k = x_j \\ 0 & \text{otherwise} \end{cases}$$