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Mathematical Topics in Kinetic Theory

Exercise Sheet 2

Exercise 5 (Maxwell–Boltzmann Functional Equation)

Let $f \in L^1(\mathbb{R}^d, \mathrm{e}^{-\gamma|x|^2}\mathrm{d}x)$ for some $\gamma \geq 0$ obey the *Maxwell–Boltzmann equation*

$$f(x)f(y) = H(|x|^2 + |y|^2, x + y) \quad \text{for a.e. } x, y \in \mathbb{R}^d \tag{MBE}$$

for some measurable function $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$.
Show that f is a Gaussian, that is, there exist constants $\alpha, A \in \mathbb{C}$ with $\operatorname{Re}\alpha < \gamma$, and $b \in \mathbb{C}^d$, such that

$$f(x) = A\mathrm{e}^{\alpha|x|^2 + b\cdot x} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

(a) Assume that $f \in \mathscr{C}^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, and that f never vanishes. If f solves (MBE), then there exist constants $\alpha, A \in \mathbb{C}$ with $\operatorname{Re}\alpha < 0$, and $b \in \mathbb{C}^d$, such that

$$f(x) = A\mathrm{e}^{\alpha|x|^2 + b\cdot x} \quad \text{for all } x \in \mathbb{R}^d.$$

Hint: Identify rotations that leave $F(x, y) = f(x)f(y)$ invariant and use this to show that $q = \log f$ satisfies the differential equation

$$0 = \partial_k q(x) - (x_k - y_k)\partial_j^2 q(y) - \partial_k q(y) + (x_j - y_j)\partial_j\partial_k q(y).$$

Conclude that there exists a constant $\alpha \in \mathbb{C}$ such that $\partial_j\partial_k q = 2\alpha\delta_{jk}$.

(b) Let g be the (L^1-)normalised centred Gaussian $g(x) = \pi^{-d/2}\,\mathrm{e}^{-|x|^2}$, and define $g_\epsilon(x) = \epsilon^{-d} g(x/\epsilon)$ for $\epsilon > 0$. Assume that $f \in L^1(\mathbb{R}^d)$ and the convolution $g_\epsilon * f$ never vanishes for all ϵ small enough. If f solves (MBE), then there exist constants $\alpha, A \in \mathbb{C}$ with $\operatorname{Re}\alpha < 0$, and $b \in \mathbb{C}^d$, such that

$$f(x) = A\mathrm{e}^{\alpha|x|^2 + b\cdot x} \quad \text{for all } x \in \mathbb{R}^d.$$

Hint: L^1-limits of Gaussians are Gaussians.

(c) If $f \in L^1(\mathbb{R}^d)$ obeys (MBE) and $f \not\equiv 0$, then for any small $\epsilon > 0$ the convolution $g_\epsilon * f$ never vanishes.

(d) If $f \in L^1(\mathbb{R}^d, \mathrm{e}^{-\gamma|x|^2}\mathrm{d}x)$ for some $\gamma \geq 0$ obeys (MBE), there exist constants $\alpha, A \in \mathbb{C}$ with $\operatorname{Re}\alpha < \gamma$, and $b \in \mathbb{C}^d$, such that

$$f(x) = A\mathrm{e}^{\alpha|x|^2 + b\cdot x} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

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