Mathematical Topics in Kinetic Theory

Exercise Sheet 3

Exercise 6 (Bobylev Identity)

Consider the (bilinear) Boltzmann collision operator with collision kernel B,

$$Q(g,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left(g(v_*')f(v') - g(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_*.$$

where the σ-representation of the collision process, in which

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \text{for } \sigma\in\mathbb{S}^2,$$

is used.

Theorem 1 (Bobylëv Identity)**.** The Fourier transform of the Boltzmann collision operator is given by

$$\widehat{Q(g,f)}(\xi) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(\xi^- + \xi_*)\hat{f}(\xi^+ - \xi_*) - \hat{g}(\xi_*)\hat{f}(\xi-\xi_*)\right)\,\mathrm{d}\xi_*\,\mathrm{d}\sigma,$$

where $\xi^\pm = \frac{\xi\pm|\xi|\sigma}{2}$ and the Fourier transform $\widehat{B}$ of B is with respect to the first argument.

Corollary 2 (Bobylëv Identity for Maxwellian Molecules)**.** Assume that the collision kernel depends only on the deviation angle, i.e. $B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) = b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)$. Then

$$\widehat{Q(g,f)}(\xi) = \int_{\mathbb{S}^{d-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(\xi^-)\hat{f}(\xi^+) - \hat{g}(0)\hat{f}(\xi)\right)\,\mathrm{d}\sigma.$$