Mathematical Topics in Kinetic Theory

Exercise Sheet 4

Exercise 7 (Symmetrisation of the Boltzmann collision kernel)

The Boltzmann collision operator with collision kernel B is given by

$$Q_B(f,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_*.$$

Show that without loss of generality, we can assume that the collision kernel B is supported on the set $0 \le \theta \le \frac{\pi}{2}$, i.e. $(v-v_*)\cdot\sigma \ge 0$.

HINT: Define

$$\overline{B}\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) := \left[B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) + B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot(-\sigma)\right)\right]1_{\{(v-v_*)\cdot\sigma\ge 0\}}$$

and show that $Q_B(f,f) = Q_{\overline{B}}(f,f)$.

SOLUTION: We simply write, with $k = \frac{v-v_*}{|v-v_*|}$,

$$\begin{aligned}
Q_B(f,f)(v) &= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, k\cdot\sigma\right)\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_* \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, k\cdot\sigma\right)\left(1_{\{(v-v_*)\cdot\sigma\ge 0\}} + 1_{\{(v-v_*)\cdot\sigma< 0\}}\right)\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_* \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, k\cdot\sigma\right)1_{\{(v-v_*)\cdot\sigma\ge 0\}}\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_* \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, k\cdot\sigma\right)1_{\{(v-v_*)\cdot\sigma< 0\}}\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_*
\end{aligned}$$

We now make a change of coordinates $\sigma \mapsto -\sigma$ on $\mathbb{S}^{d-1}$. Notice that this change of coordinates is involutive with unit Jacobian, and maps $v' \mapsto v_*'$, $v_*' \mapsto v'$. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, k\cdot\sigma\right)1_{\{(v-v_*)\cdot\sigma< 0\}}\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_* \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, -k\cdot\sigma\right)1_{\{-(v-v_*)\cdot\sigma< 0\}}\left(f(v')f(v_*') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_* \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, -k\cdot\sigma\right)1_{\{(v-v_*)\cdot\sigma> 0\}}\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_*.
\end{aligned}$$

Putting the two expressions back together ($\{(v-v_*)\cdot\sigma = 0\}$ is a set of measure zero) we get the equality

$$Q_B(f,f) = Q_{\overline{B}}(f,f)$$

with $\overline{B}\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) := \left[B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) + B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot(-\sigma)\right)\right]1_{\{(v-v_*)\cdot\sigma\ge 0\}}$ supported on $\{(v-v_*)\cdot\sigma \ge 0\}$.

Exercise 8 (Momentum transfer and weak formulation for singular collision kernels)

Assume that $B(|z|, k\cdot\sigma) = |z|^\gamma b(\cos\theta)$, where $k = \frac{z}{|z|}$, $\cos\theta = k\cdot\sigma$, $\theta \in [0,\pi/2]$ (see Exercise above), and angular kernel given by

$$\sin^{d-2}\theta\, b(\cos\theta) = \frac{\kappa}{(\sin\theta)^{1+2\nu}}$$

for some $\kappa > 0$, $0 \le \nu < 1$, in dimensions $d \ge 2$.

(a) Show that for all $k \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)\,\mathrm{d}\sigma = \infty,$$

that is, $b \notin L^1([0,1],(1-u^2)^{\frac{d-3}{2}}\,\mathrm{d}u)$, but that

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma < \infty.$$

(b) Define the *cross section for momentum transfer* by[1]

$$\mathcal{M}(|z|) := \int_{\mathbb{S}^{d-1}} B(|z|, k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma, \quad z \in \mathbb{R}^d.$$

Show that $z \mapsto \mathcal{M}(|z|) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only of $\gamma > -d$.

(c) Let $\mathcal{T}$ be the linear operator

$$\mathcal{T} : \varphi \mapsto \int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)\,(\varphi(v') - \varphi(v))\,\mathrm{d}\sigma, \quad k = \frac{v-v_*}{|v-v_*|}.$$

Show that for all $\varphi \in W^{2,\infty}(\mathbb{R}^d)$,

$$|(\mathcal{T}\varphi)(v,v_*)| \le \frac{1}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}|v-v_*|\left(1 + \frac{|v-v_*|}{2}\right)\mathcal{M}(|v-v_*|).$$

(d) Conclude that if $\gamma \ge -1$ and $f, g \in L^1_{2+\gamma}(\mathbb{R}^d) = \left\{f \in L^1(\mathbb{R}^d) : \|f\|_{L^1_{2+\gamma}} := \int_{\mathbb{R}^d} |f(v)|\langle v\rangle^{2+\gamma}\,\mathrm{d}v < \infty\right\}$, where $\langle v\rangle = (1+|v|^2)^{1/2}$, then Maxwell's weak formulation

$$\langle Q(g,f),\varphi\rangle = \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} |v-v_*|^\gamma b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,(\varphi(v') - \varphi(v))\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

is well-definied for all $\varphi \in W^{2\infty}(\mathbb{R}^d)$, with the bound

$$|\langle Q(g,f),\varphi\rangle| \le C_b\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\|g\|_{L^1_{2+\gamma}}\|f\|_{L^1_{2+\gamma}}.$$

[1] So that $\int_{\mathbb{S}^{d-1}} B(|z|, k\cdot\sigma)(v'-v)\,\mathrm{d}\sigma = \frac{1}{2}(v-v_*)\mathcal{M}(|v-v_*|)$, see also the calculations for part (c).

Solution:

(a) We use spherical coordinates on the sphere such that k points to the north pole to obtain

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)\,\mathrm{d}\sigma = |\mathbb{S}^{d-2}|\int_0^{\pi/2} b(\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta = \kappa|\mathbb{S}^{d-2}|\int_0^{\pi/2}(\sin\theta)^{-1-2\nu}\,\mathrm{d}\nu = \infty$$

since the integrand behaves like the non-integrable function $\theta^{-1-2\nu}$ near zero ($-1-2\nu \le -1$). In other words, since

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)\,\mathrm{d}\sigma = |\mathbb{S}^{d-2}|\int_0^{\pi/2} b(\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta = \int_0^1 b(u)(1-u^2)^{\frac{d-3}{2}}\,\mathrm{d}u,$$

we have $b \notin L^1([0,1],(1-u^2)^{\frac{d-3}{2}}\,\mathrm{d}u)$.
A similar computation yields (with $1-\cos\theta = 2\sin^2\frac{\theta}{2}$)

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma &= |\mathbb{S}^{d-2}|\int_0^{\pi/2} b(\cos\theta)(1-\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta\\ &= 2\kappa|\mathbb{S}^{d-2}|\int_0^{\pi/2}\frac{\sin^2\frac{\theta}{2}}{(\sin\theta)^{1+2\nu}}\,\mathrm{d}\theta =: C_b < \infty\end{aligned}$$

since the singularity at 0 of order $1+2\nu$ is made integrable by the additional factor θ^2 from $\sin^2\frac{\theta}{2}$.

(b) In our example $B(|z|,k\cdot\sigma) = |z|^\gamma b(k\cdot\sigma)$, so that the cross-section for momentum transfer is given by

$$\mathcal{M}(|z|) = \int_{\mathbb{S}^{d-1}} |z|^\gamma b(k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma = |z|^\gamma\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma = C_b|z|^\gamma,$$

where C_b is the constant from exercise (a). Since the function $z\mapsto|z|^\gamma$ is locally integrable if and only if $\gamma > -d$, the claim follows.

(c) We use Taylor's formula (2^{nd} order) to obtain

$$\varphi(v') = \varphi(v) + \nabla\varphi(v)\cdot(v'-v) + \int_0^1 \big\langle v'-v, D^2\varphi(v+\tau(v'-v))(v'-v)\big\rangle\,(1-\tau)\,\mathrm{d}\tau$$

and can therefore estimate

$$\begin{aligned}|(\mathcal{T}\varphi)(v,v_*)| \le{}& \left|\int_{\mathbb{S}^{d-1}} B(|v-v_*|,k\cdot\sigma)\nabla\varphi(v)\cdot(v'-v)\,\mathrm{d}\sigma\right|\\ &+\int_{\mathbb{S}^{d-1}} B(|v-v_*|,k\cdot\sigma)\int_0^1\left|\big\langle v'-v, D^2\varphi(v+\tau(v'-v))(v'-v)\big\rangle\right|(1-\tau)\,\mathrm{d}\tau.\end{aligned}$$

For the first integral, we have

$$\int_{\mathbb{S}^{d-1}} B(|v-v_*|,k\cdot\sigma)\nabla\varphi(v)\cdot(v'-v)\,\mathrm{d}\sigma = \nabla\varphi(v)\cdot\int_{\mathbb{S}^{d-1}} B(|v-v_*|,k\cdot\sigma)\,(v'-v)\,\mathrm{d}\sigma.$$

Notice that $v'-v = -\frac{v-v_*}{2} + \frac{|v-v_*|}{2}\sigma = -\frac{|v-v_*|}{2}(k-\sigma)$, and if we represent $\sigma\in\mathbb{S}^{d-1}$ as

$$\sigma = (k\cdot\sigma)k + \sqrt{1-k\cdot\sigma}\,\omega,$$

where $\omega = \frac{(1-kk^T)\sigma}{|(1-kk^T)\sigma|} = \frac{\sigma-(k\cdot\sigma)k}{\sqrt{1-k\cdot\sigma}}$ is the unit vector in the direction of the projection of σ onto the hyperplane orthogonal to k, we obtain

$$v' - v = -\frac{|v-v_*|}{2}(1-k\cdot\sigma)k + \frac{|v-v_*|}{2}\sqrt{1-k\cdot\sigma}\,\omega.$$

In particular, by symmetry (rotations of the unit sphere keeping the k direction fixed), the second contribution proportional to ω vanishes when integrating over $\mathbb{S}^{d-1}$, and we get

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)\,(v'-v)\,\mathrm{d}\sigma &= -\frac{|v-v_*|}{2}k\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma \\ &= -\frac{|v-v_*|}{2}k\mathcal{M}(|v-v_*|).\end{aligned}$$

It follows that

$$\left|\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)\nabla\varphi(v)\cdot(v'-v)\,\mathrm{d}\sigma\right| \le \frac{1}{2}|v-v_*|\mathcal{M}(|v-v_*|)\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}.$$

For the second integral we estimate

$$\left|\left\langle v'-v, D^2\varphi(v+\tau(v'-v))(v'-v)\right\rangle\right| \le |v'-v|^2\|D^2\varphi\|_{L^\infty(\mathbb{R}^d)} = \frac{|v-v_*|^2}{2}(1-k\cdot\sigma)\|D^2\varphi\|_{L^\infty(\mathbb{R}^d)},$$

where we used that

$$|v'-v|^2 = \left|-\frac{|v-v_*|}{2}(k-\sigma)\right|^2 = \frac{|v-v_*|^2}{4}(k-\sigma)^2 = \frac{|v-v_*|^2}{2}(1-k\cdot\sigma).$$

Hence

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)\int_0^1 \left|\left\langle v'-v, D^2\varphi(v+\tau(v'-v))(v'-v)\right\rangle\right|(1-\tau)\,\mathrm{d}\tau \\ &\quad\le \frac{|v-v_*|^2}{2}\|D^2\varphi\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma\int_0^1(1-\tau)\,\mathrm{d}\tau \\ &\quad= \frac{|v-v_*|^2}{4}\|D^2\varphi\|_{L^\infty(\mathbb{R}^d)}\mathcal{M}(|v-v_*|).\end{aligned}$$

Putting the two estimates together, we obtain with

$$\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}, \|D^2\varphi\|_{L^\infty(\mathbb{R}^d)} \le \|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)} = \left(\sum_{|\alpha|\le 2}\|\partial^\alpha\varphi\|^2_{L^\infty(\mathbb{R}^d)}\right)^{1/2}$$

the final estimate

$$|(\mathcal{T}\varphi)(v,v_*)| \le \frac{1}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}|v-v_*|\left(1+\frac{|v-v_*|}{2}\right)\mathcal{M}(|v-v_*|).$$

(d) We can express Maxwell's weak formulation of the Boltzmann operator in terms of the operator $\mathcal{T}$ and use the estimate derived in part (c):

$$\begin{aligned}\left\langle Q(g,f),\varphi\right\rangle &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)g(v_*)f(v)(\varphi(v')-\varphi(v))\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\ &= \int_{\mathbb{R}^{2d}} g(v_*)f(v)\int_{\mathbb{S}^{d-1}} B(|v-v_*|, k\cdot\sigma)(\varphi(v')-\varphi(v))\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_* \\ &= \int_{\mathbb{R}^{2d}} g(v_*)f(v)(\mathcal{T}\varphi)(v,v_*)\,\mathrm{d}v\,\mathrm{d}v_*,\end{aligned}$$

so

$$\begin{aligned}\left|\left\langle Q(g,f),\varphi\right\rangle\right| &\le \int_{\mathbb{R}^{2d}} g(v_*)f(v)|(\mathcal{T}\varphi)(v,v_*)|\,\mathrm{d}v\,\mathrm{d}v_* \\ &\le \frac{1}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\int_{\mathbb{R}^{2d}} g(v_*)f(v)|v-v_*|\left(1+\frac{|v-v_*|}{2}\right)\mathcal{M}(|v-v_*|)\,\mathrm{d}v\,\mathrm{d}v_* \\ &= \frac{C_b}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\int_{\mathbb{R}^{2d}} g(v_*)f(v)|v-v_*|^{1+\gamma}\left(1+\frac{|v-v_*|}{2}\right)\mathrm{d}v\,\mathrm{d}v_*.\end{aligned}$$

Since $\gamma \ge -1$, we have

$$|v-v_*|^{1+\gamma}\left(1+\frac{|v-v_*|}{2}\right) \le 2^{1+\gamma}\left(1+\frac{|v-v_*|}{2}\right)^{2+\gamma}$$

and by convexity (the function $t \mapsto (1+t)^p$, $p \ge 1$, is convex) this can be bounded by

$$2^{1+\gamma}\left(1+\frac{|v-v_*|}{2}\right)^{2+\gamma} \le 2^{\gamma}\left[(1+|v|)^{2+\gamma}+(1+|v_*|)^{2+\gamma}\right].$$

Finally, using the simple estimate $1+|v| \le \sqrt{2}(1+|v|^2)^{1/2} = \sqrt{2}\langle v\rangle$, we obtain

$$|v-v_*|^{1+\gamma}\left(1+\frac{|v-v_*|}{2}\right) \le 2^{\gamma}2^{\frac{2+\gamma}{2}}\left(\langle v\rangle^{2+\gamma}+\langle v_*\rangle^{2+\gamma}\right).$$

It follows that

$$\begin{aligned}\left|\left\langle Q(g,f),\varphi\right\rangle\right| &\le \frac{c_{b,\gamma}}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\int_{\mathbb{R}^{2d}} g(v_*)f(v)\left(\langle v\rangle^{2+\gamma}+\langle v_*\rangle^{2+\gamma}\right)\mathrm{d}v\,\mathrm{d}v_* \\ &= \frac{c_{b,\gamma}}{2}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\left(\|g\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1_{2+\gamma}(\mathbb{R}^d)}+\|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^1_{2+\gamma}(\mathbb{R}^d)}\right) \\ &\le c_{b,\gamma}\|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)}\|f\|_{L^1_{2+\gamma}(\mathbb{R}^d)}\|g\|_{L^1_{2+\gamma}(\mathbb{R}^d)},\end{aligned}$$

in particular that the Boltzmann operator is well-defined in the sense of Maxwell's weak formulation (in the dual space of $W^{2,\infty}(\mathbb{R}^d)$).