Mathematical Topics in Kinetic Theory

Exercise Sheet 5

Exercise 9 (Velocity averaging lemma)

Theorem 1. Let $f \in L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$ be a (weak) solution of the inhomogeneous transport equation

$$\partial_t f + v \cdot \nabla_x f = g$$

with inhomogeneity $g \in L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$, and let $\varphi \in L^\infty(\mathbb{R}^d_v)$ have compact support. Then the *velocity average* $m : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}$,

$$m(t,x) := \int_{\mathbb{R}^d_v} f(t,x,v)\,\varphi(v)\,\mathrm{d}v$$

satisfies $m \in L^2(\mathbb{R}_t; H^{1/2}(\mathbb{R}^d_x))$ and

$$\|m\|_{L^2(\mathbb{R}_t;H^{1/2}(\mathbb{R}^d_x))} \leq C(\varphi)\|f\|^{1/2}_{L^2_{t,x,v}}\|g\|^{1/2}_{L^2_{t,x,v}},$$

with a constant $C(\varphi)$ depending only on the support $\operatorname{supp}\varphi$ of φ and $\|\varphi\|_{L^\infty_v}$.

Remark 2. Here we use

$$\|u\|_{H^s(\mathbb{R}^d_x)} = \||\cdot|^s\widehat{u}\|_{L^2(\mathbb{R}^d_\xi)}, \quad s > 0,$$

for a function $u \in H^s(\mathbb{R}^d_x) = \{v \in L^2(\mathbb{R}^d_x) : |\cdot|^s\widehat{v} \in L^2(\mathbb{R}^d_\xi)\}$, the (fractional) Sobolev space of order $s > 0$, where

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}\,\mathrm{d}x$$

is the Fourier transform of u.