Mathematical Topics in Kinetic Theory

Exercise Sheet 5

Exercise 9 (Velocity averaging lemma)

Theorem 1. Let $f \in L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$ be a (weak) solution of the inhomogeneous transport equation

$$\partial_t f + v \cdot \nabla_x f = g$$

with inhomogeneity $g \in L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$, and let $\varphi \in L^\infty(\mathbb{R}^d_v)$ have compact support. Then the *velocity average* $m : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}$,

$$m(t,x) := \int_{\mathbb{R}^d_v} f(t,x,v)\, \varphi(v)\, \mathrm{d}v$$

satisfies $m \in L^2(\mathbb{R}_t; H^{1/2}(\mathbb{R}^d_x))$ and

$$\|m\|_{L^2(\mathbb{R}_t; H^{1/2}(\mathbb{R}^d_x))} \leq C(\varphi) \|f\|^{1/2}_{L^2_{t,x,v}} \|g\|^{1/2}_{L^2_{t,x,v}},$$

with a constant $C(\varphi)$ depending only on the support $\operatorname{supp}\varphi$ of φ and $\|\varphi\|_{L^\infty_v}$.

Remark 2. Here we use

$$\|u\|_{H^s(\mathbb{R}^d_x)} = \| |\cdot|^s \widehat{u} \|_{L^2(\mathbb{R}^d_\xi)}, \quad s > 0,$$

for a function $u \in H^s(\mathbb{R}^d_x) = \{v \in L^2(\mathbb{R}^d_x) : |\cdot|^s \widehat{v} \in L^2(\mathbb{R}^d_\xi)\}$, the (fractional) Sobolev space of order $s > 0$, where

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x)\, \mathrm{e}^{-2\pi \mathrm{i} x \cdot \xi}\, \mathrm{d}x$$

is the Fourier transform of u.

Proof. Let

$$\widehat{f}(\tau, \xi, v) = \int_{\mathbb{R}_t} \int_{\mathbb{R}^d_x} f(t,x,v)\, \mathrm{e}^{-2\pi \mathrm{i}(t\tau + x\cdot\xi)}\, \mathrm{d}x\, \mathrm{d}t$$

be the partial Fourier transform of f with respect to time and space variables t, x, and define $\widehat{g}$ analogously. Both are well-defined since f and g are square integrable functions in all three variables $(t,x,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. Taking the Fourier transform of the transport equation yields the equivalent equation

$$2\pi \mathrm{i}(\tau + v \cdot \xi)\widehat{f}(\tau, \xi, v) = \widehat{g}(\tau, \xi, v).$$

Remark 3. Strictly speaking, since the solutions are only weak, one has take the weak formulation of the equation: $f \in L^2_{t,x,v}$ is a weak solution of the transport equation if

$$-\int_{\mathbb{R}^{1+2d}_{t,x,v}} f(t,x,v)\,(\partial_t \phi(t,x,v) + v \cdot \nabla_x \phi(t,x,v))\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{1+2d}_{t,x,v}} g(t,x,v)\phi(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$

for all compactly supported smooth test functions $\phi \in \mathscr{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$. By Plancherel (unitarity of the Fourier transform) and properties of the Fourier transform under differentiation, this is equivalent to

$$2\pi\mathrm{i} \int_{\mathbb{R}^{1+2d}_{\tau,\xi,v}} \widehat{f}(\tau,\xi,v)\,(\tau + v \cdot \xi)\,\widehat{\phi}(\tau,\xi,v)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}v = \int_{\mathbb{R}^{1+2d}_{\tau,\xi,v}} \widehat{g}(\tau,\xi,v)\widehat{\phi}(\tau,\xi,v)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}v$$

for all $\phi \in \mathscr{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$. Since the set $\mathscr{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$ is dense in $L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$, this uniquely identifies

$$2\pi\mathrm{i}(\tau + v \cdot \xi)\widehat{f}(\tau,\xi,v) = \widehat{g}(\tau,\xi,v) \quad \text{in} \quad L^2(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v).$$

In particular, we have

$$\widehat{f}(\tau,\xi,v) = \frac{\widehat{g}(\tau,\xi,v)}{2\pi\mathrm{i}(\tau + v \cdot \xi)}$$

for all (τ,ξ,v) such that $\tau + v \cdot \xi \neq 0$.
We can use this information to estimate the Fourier transform of m,

$$\begin{aligned}\widehat{m}(\tau,\xi) &= \int_{\mathbb{R}_t \times \mathbb{R}^d_x} m(t,x)\,\mathrm{e}^{-2\pi\mathrm{i}(t\tau + x\cdot\xi)}\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v} f(t,x,v)\varphi(v)\,\mathrm{e}^{-2\pi\mathrm{i}(t\tau + x\cdot\xi)}\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d_v} \widehat{f}(\tau,\xi,v)\varphi(v)\,\mathrm{d}v.\end{aligned}$$

By the triangle inequality, we obtain for any parameter $\alpha > 0$ (to be chosen later),

$$\begin{aligned}|\widehat{m}(\tau,\xi)| &\leq \int_{\{|\tau+v\cdot\xi|<\alpha\}} \left|\widehat{f}(\tau,\xi,v)\right| |\varphi(v)|\,\mathrm{d}v + \int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \left|\widehat{f}(\tau,\xi,v)\right| |\varphi(v)|\,\mathrm{d}v \\ &= \int_{\{|\tau+v\cdot\xi|<\alpha\}} \left|\widehat{f}(\tau,\xi,v)\right| |\varphi(v)|\,\mathrm{d}v + \frac{1}{2\pi}\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \frac{\left|\widehat{g}(\tau,\xi,v)\right|}{|\tau + v\cdot\xi|} |\varphi(v)|\,\mathrm{d}v \\ &\leq \left(\int_{\{|\tau+v\cdot\xi|<\alpha\}} \left|\widehat{f}(\tau,\xi,v)\right|^2 \mathrm{d}v\right)^{1/2} \left(\int_{\{|\tau+v\cdot\xi|<\alpha\}} |\varphi(v)|^2\,\mathrm{d}v\right)^{1/2} \\ &\quad + \frac{1}{2\pi}\left(\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \left|\widehat{g}(\tau,\xi,v)\right|^2 \mathrm{d}v\right)^{1/2} \left(\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \frac{|\phi(v)|^2}{|\tau + v\cdot\xi|^2}\,\mathrm{d}v\right)^{1/2}\end{aligned}$$

where the last inequality is the Cauchy-Schwartz inequality. The integrals involving f and g can be further estimated by

$$\int_{\{|\tau+v\cdot\xi|<\alpha\}} \left|\widehat{f}(\tau,\xi,v)\right|^2 \mathrm{d}v \leq \int_{\mathbb{R}^d_v} \left|\widehat{f}(\tau,\xi,v)\right|^2 \mathrm{d}v = \left\|\widehat{f}(\tau,\xi,\cdot)\right\|^2_{L^2_v}$$

and, correspondingly,

$$\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \left|\widehat{g}(\tau,\xi,v)\right|^2 \mathrm{d}v \leq \int_{\mathbb{R}^d_v} \left|\widehat{g}(\tau,\xi,v)\right|^2 \mathrm{d}v = \left\|\widehat{g}(\tau,\xi,\cdot)\right\|^2_{L^2_v},$$

the partial L^2-norm with respect to the velocity variable.
Let us analyse the integrals involving the function φ better. In the first one, $\int_{\{|\tau+v\cdot\xi|<\alpha\}} |\varphi(v)|^2 \, \mathrm{d}v$, the region of integration $\{v \in \mathbb{R}^d : |\tau + v\cdot\xi| < \alpha\}$ is the region between two parallel planes $v \cdot \frac{\xi}{|\xi|} = \frac{\pm\alpha-\tau}{|\xi|}$ orthogonal to the direction ξ with distance $\frac{2\alpha}{|\xi|}$. Since the support of φ is compact and φ bounded, we can estimate

$$\int_{\{|\tau+v\cdot\xi|<\alpha\}} |\varphi(v)|^2 \, \mathrm{d}v \leq \|\varphi\|^2_{L^\infty(\mathbb{R}^d_v)} \left|\{v \in \mathbb{R}^d : |\tau + v\cdot\xi| < \alpha\} \cap \operatorname{supp}\varphi\right| \leq \frac{C(\varphi)\alpha}{|\xi|},$$

with a constant depending only on $\operatorname{supp}\varphi$ and $\|\varphi\|_{L^\infty}$.
For the second integral, $\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \frac{|\phi(v)|^2}{|\tau+v\cdot\xi|^2} \, \mathrm{d}v$, we do a change of coordinates $v \mapsto (y, v^\perp)$, $y \in \mathbb{R}$, $v^\perp \perp \xi$,

$$v = \left(y - \frac{\tau}{|\xi|}\right)\frac{\xi}{|\xi|} + v^\perp.$$

Then $\mathrm{d}v = \mathrm{d}y \, \mathrm{d}v^\perp$ and $\tau + v\cdot\xi = \tau + \left(y - \frac{\tau}{|\xi|}\right)|\xi| = y|\xi|$ and

$$\begin{aligned}\int_{\{|\tau+v\cdot\xi|\geq\alpha\}} \frac{|\phi(v)|^2}{|\tau+v\cdot\xi|^2} \, \mathrm{d}v &= \int_{|y|\geq\alpha/|\xi|} \mathrm{d}y \int_{\mathbb{R}^{d-1}} \mathrm{d}v^\perp \frac{|\varphi(v(y,v^\perp))|^2}{y^2|\xi|^2} \\ &\leq \|\varphi\|^2_{L^\infty} |\operatorname{supp}\varphi| \int_{|y|\geq\alpha/|\xi|} \frac{1}{y^2|\xi|^2} \, \mathrm{d}y = \frac{C(\varphi)}{\alpha|\xi|},\end{aligned}$$

with (another) constant $C(\varphi)$ depending only on $\operatorname{supp}\varphi$ and $\|\varphi\|_{L^\infty}$.
Putting the estimates obtained so far together, we have the bound

$$|\widehat{m}(\tau,\xi)| \leq \frac{C(\varphi)}{|\xi|^{1/2}} \left(\alpha^{1/2}\|\widehat{f}(\tau,\xi,\cdot)\|_{L^2_v} + \alpha^{-1/2}\|\widehat{g}(\tau,\xi,\cdot)\|_{L^2_v}\right).$$

In particular,

$$\|m\|^2_{L^2(\mathbb{R}_t;H^{1/2}(\mathbb{R}^d_x))} = \int_{\mathbb{R}} \|m(t,\cdot)\|^2_{H^{1/2}(\mathbb{R}^d_x)} \, \mathrm{d}t = \int_{\mathbb{R}_\tau\times\mathbb{R}^d_\xi} |\xi| \, |\widehat{m}(\tau,\xi)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi$$

Using the bound on $\widehat{m}$ and the simple inequality $(a+b)^2 \leq 2(a^2+b^2)$ for any $a, b \in \mathbb{R}$, we can estimate

$$\begin{aligned}\|m\|^2_{L^2(\mathbb{R}_t;H^{1/2}(\mathbb{R}^d_x))} &\leq C(\varphi)^2 \int_{\mathbb{R}_\tau\times\mathbb{R}^d_\xi} \left(\alpha^{1/2}\|\widehat{f}(\tau,\xi,\cdot)\|_{L^2_v} + \alpha^{-1/2}\|\widehat{g}(\tau,\xi,\cdot)\|_{L^2_v}\right)^2 \mathrm{d}\tau \, \mathrm{d}\xi \\ &\leq 2C(\varphi)^2 \int_{\mathbb{R}_\tau\times\mathbb{R}^d_\xi} \left(\alpha\|\widehat{f}(\tau,\xi,\cdot)\|^2_{L^2_v} + \alpha^{-1}\|\widehat{g}(\tau,\xi,\cdot)\|^2_{L^2_v}\right) \mathrm{d}\tau \, \mathrm{d}\xi \\ &= 2C(\varphi)^2 \left(\alpha \int_{\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^d_v} |\widehat{f}(\tau,\xi,v)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi \, \mathrm{d}v + \alpha^{-1} \int_{\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^d_v} |\widehat{g}(\tau,\xi,v)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi \, \mathrm{d}v\right) \\ &= 2C(\varphi)^2 \left(\alpha\|\widehat{f}\|^2_{L^2_{\tau,\xi,v}} + \alpha^{-1}\|\widehat{g}\|^2_{L^2_{\tau,\xi,v}}\right) = 2C(\varphi)^2 \left(\alpha\|f\|^2_{L^2_{t,x,v}} + \alpha^{-1}\|g\|^2_{L^2_{t,x,v}}\right)\end{aligned}$$

Choosing $\alpha = \frac{\|g\|_{L^2_{t,x,v}}}{\|f\|_{L^2_{t,x,v}}} > 0$ then yields

$$\|m\|^2_{L^2(\mathbb{R}_t;H^{1/2}(\mathbb{R}^d_x))} \leq 4C(\varphi)^2\|f\|_{L^2_{t,x,v}}\|g\|_{L^2_{t,x,v}}$$

and taking square roots completes the proof. □