Mathematical Topics in Kinetic Theory

Exercise Sheet 7

Exercise 12 (Poincaré limit)

(a) Let $F \in \mathscr{C}(\mathbb{R}^N)$ and let $\mathrm{d}\sigma_R^N$ denote the uniform probability measure on $\mathbb{S}^{N-1}(R)$. Then

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}(R)} F \,\mathrm{d}\sigma_R^N \\
&= \frac{1}{|\mathbb{S}^{N-1}| R^{N-2}} \sum_{\tau\in\{+,-\}} \int_{\mathbb{R}^{N-1}} F\left(v_1,\dots,v_{N-1},\tau\sqrt{R^2-\sum_{i=1}^{N-1} v_i^2}\right)\left(R^2-\sum_{i=1}^{N-1} v_i^2\right)_+^{-1/2} \mathrm{d}v_1\dots\mathrm{d}v_{N-1}
\end{aligned}$$

(b) If $F \in \mathscr{C}(\mathbb{R}^N)$ and $\varphi \in \mathscr{C}_b(\mathbb{R}^k)$, $1 \le k \le N-1$, then

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}(R)} \varphi(v_1,\dots,v_k)\, F(v_1,\dots,v_N)\,\mathrm{d}\sigma_R^N \\
&= \frac{|\mathbb{S}^{N-k-1}|}{|\mathbb{S}^{N-1}| R^{N-2}} \int_{\mathbb{R}^k} \varphi(v_1,\dots,v_k)\left(R^2-\sum_{i=1}^{k} v_i^2\right)_+^{\frac{N-k-2}{2}} \\
&\qquad\times\left(\int_{\mathbb{S}^{N-k-1}\left(\sqrt{R^2-\sum_{i=1}^k v_i^2}\right)} F\,\mathrm{d}\sigma^{N-k}_{\sqrt{R^2-\sum_{i=1}^k v_i^2}}\right)\mathrm{d}v_1\dots\mathrm{d}v_k
\end{aligned}$$

(c) Let $\varphi \in \mathscr{C}_b(\mathbb{R}^k)$ and $\mathrm{d}\sigma^N := \mathrm{d}\sigma^N_{\sqrt{N}}$. Show that

$$\lim_{N\to\infty} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \varphi(v_1,\dots,v_k)\,\mathrm{d}\sigma^N = (2\pi)^{-k/2} \int_{\mathbb{R}^k} \varphi(v_1,\dots,v_k)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^k v_i^2}\,\mathrm{d}v_1\dots\mathrm{d}v_k.$$

Hint: Use Stirling's formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{\mathrm{e}}\right)^z\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \quad z > 0.$$