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SS 2011

Mathematical Methods of Quantum Mechanics
1st Exercise Sheet

1. Let H be a Hilbert space. Prove the following statements:

(i) A sequence (u_n) in H converges to $u \in H$ if and only if

$$(*)\ \lim_{n\to\infty} \langle u_n, \phi\rangle = \langle u, \phi\rangle \quad (\phi \in H)$$

and $\lim_{n\to\infty} \|u_n\| = \|u\|$. Give an example to show that the condition (*) alone does not imply convergence of (u_n) in H.

(ii) If M is a subspace of H, then

$$(M^\perp)^\perp = \overline{M}.$$

2. Let A, B be bounded operators on the Hilbert space H. Show that

(i) $(A+B)^* = A^* + B^*, (\lambda A)^* = \overline{\lambda} A^*$ for $\lambda \in \mathbb{C}$.
(ii) $(AB)^* = B^* A^*$
(iii) $\|A\| = \|A^*\|$
(iv) $A^{**} = A$
(v) $\|AA^*\| = \|A^*A\| = \|A\|^2$
(vi) $\ker A = (\operatorname{ran} A^*)^\perp, \ker A^* = (\operatorname{ran} A)^\perp$.

3. Let

$$D := \text{linear hull of} \{e^{-|x|^2/2} x_1^{k_1} x_2^{k_2} x_3^{k_3} : k_i \in \mathbb{N}_0\} \subset L^2(\mathbb{R}^3).$$

(i) Show that $x \mapsto \phi(x) e^{-|x|^2/2} \in L^1(\mathbb{R}^3)$ for any $\phi \in L^2(\mathbb{R}^3)$.

(ii) Let $\phi \in L^2(\mathbb{R}^3)$ be such that $\int_{\mathbb{R}^3} \phi(x) e^{-|x|^2/2} x_1^{k_1} x_2^{k_2} x_3^{k_3} dx = 0$ for all $(k_1, k_2, k_3) \in \mathbb{N}_0^3$. Prove that

$$\int_{\mathbb{R}^3} \phi(x) e^{-|x|^2/2} e^{i\xi\cdot x} dx = 0 \quad (\xi \in \mathbb{R}^3)$$

holds.
Hint: use the usual series $e^z = \sum_{n=0}^\infty \frac{1}{n!} z^n$ for the exponential function.

(iii) Show that D is dense in $L^2(\mathbb{R}^3)$.
Hint: Fourier transform.